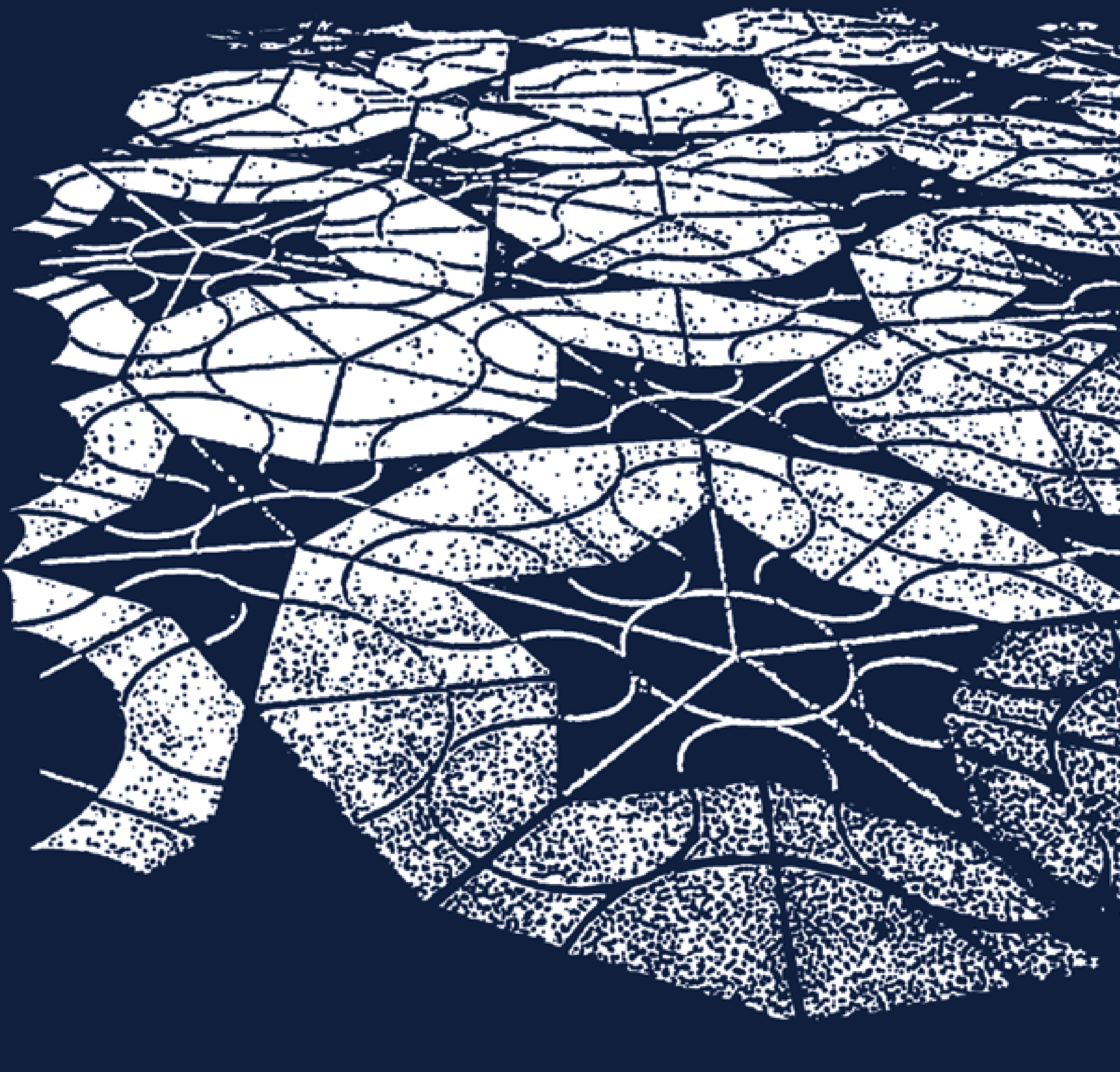
Copernicus Center Reports, no. 2



Copernicus Center Reports »

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Copernicus Center Reports no. 2

Kraków 2011

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The second volume of Copernicus Center Reports provides a summary of the Center's activities in 2010. It was a year of the institution's further consolidation: new projects were initiated and new collaborations established, featuring several of the hallmarks of the Center's activity. The Kraków Methodological Conferences garnered international appeal and the 2010 honorary guest of the conference was Sir Roger Penrose, who delivered the 2010 Copernicus Center Lecture. The Conference gathered together leading scholars in cosmology and philosophy of physics. In 2010 the Copernicus Center Press was also established, as an imprint of the Konsorcjum Akademickie Publishing House: we have already published four monographic books, and plan to publish around ten books a year. Equally important are other events and initiatives of the Center: a number of conferences and seminars, often with the participation of leading researchers from all over the world; the constant development of the Copernicus Center Website; the series of lectures organized within the Copernicus Center College, and so on.

However, the dates and figures tell only half of the story as there is no research institution without serious research activities. Thus, we are happy to see a growing body of results achieved by the eleven research teams affiliated with the Center. All of this activity would be impossible without the generous help and hard work of many individuals and institutions, whom we would like to sincerely thank.

In addition to the reporting section, we include two papers. One of them was written for the 2010 Kraków Methodological Conference by the late Archbishop Józef Życiński. His sudden death has left a void in the Polish intellectual and spiritual landscape. Fortunately, his writings remain, and one may hope that they will serve as a source of inspiration and a solid partner in the dialogue between science and religion.

Bartosz Brożek

Annual Report 2010 »

Research

Within the Copernicus Center there are 11 established research groups.

I. Copernican Group

Head

- Professor Michał Kokowski (Institute of the History of Science, PAN)

Members

- Professor Tadeusz Sierotowicz (John Paul II Pontifical University)
- Professor Jarosław Włodarczyk (Institute of the History of Science, PAN)

Research fields

- Copernicus (1473-1543) against the backdrop of his times (biographical and historical aspects)
- detailed analysis of Copernicus' achievements from scientific and cultural perspectives
- detailed analysis of the genesis and reception of Copernicus' achievements
- detailed analysis of the theories formulated by the advocates of Copernicanism (Gallileo, Kepler)

Recent activities

- In 2009 Professor Tadeusz Sierotowicz published a Polish translation of Galileo's *Il Saggiatore – Waga probiercza*, OBI-Copernicus Center, Kraków – Tarnów 2009.

- Professor Tadeusz Sierotowicz has prepared three papers pertaining to *Il Saggiatore*, to be published in 2011.
- Professor Tadeusz Sierotowicz is preparing a Polish translation of Galileo's letters on sunspots. The translation, together with a commentary, will be ready in 2013.
- Professor Jaroslaw Włodarczyk co-edited the book monograph: Jerzy Dobrzycki, *Selected Papers on Medieval and Renaissance Astronomy* (Studia Copernicana, vol. XLIII), J. Włodarczyk, R. L. Kremer (eds.), Warszawa 2010.
- Professor Michał Kokowski was the initiator and the head organizer of the conference „Tajemnica grobu Mikołaja Kopernika. Dialog ekspertów" (The Nicholas Copernicus grave mystery. A dialogue of experts). The conference took place on February 22-23, 2010 in Kraków. See the description in the conferences section of the report.
- In 2010 Professor Michał Kokowski was awarded the Nicholas Copernicus Prize of the Foundation of the City of Kraków (awarded every five years by the Polish Academy of Arts and Sciences) for the book monograph *Różne oblicza Mikołaja Kopernika. Spotkania z historią interpretacji* (The Many Faces of Nicolaus Copernicus. Encounters with the History of Interpretation), Warszawa: Instytut Historii Nauki PAN, Kraków: Polska Akademia Umiejętności, 2009.

II. Science and Religion

Head

- Fr. Dr. Zbigniew Liana (John Paul II Pontifical University)

Members

- Sr. Dr. Teresa Obolevitch (John Paul II Pontifical University)
- Dr. Jacek Rodzeń (John Paul II Pontifical University)

Research Fields

- the relationship between science and religion in the 20th century (in cooperation with the PAU *Fides et Ratio* commission)
- the question of science-faith in the life and work of Karol Wojtyła – John Paul II (in cooperation with the PAU *Fides et Ratio* commission)
- history of the relationship between science-religion
- the relationship between technology and religion
- the problems of the relationship between science and religion in Russian philosophy
- epistemological and ontological questions in the context of the relationship between science and religion
- the problematic of symbolism in patristic thought and in Russian philosophy

Recent activities

In 2010, the research team, in cooperation with the *Fides et Ratio* Commission of the Polish Academy of Arts and Sciences, organized the following lectures:

- 14.01.2010: Michał Heller, *Teologia naturalna a współczesna kosmologia* (Natural Theology and the Contemporary Cosmology)
- 18.02.2010: Jerzy Dadaczyński, *Problem sprzeczności w teologii. Jakiej logiki potrzebuje współczesna teologia?* (The Problem of Contradiction in Theology: What Logic for Contemporary Theology?)
- 18.03.2010: Stanisław Wszołek, *Zaniedbany argument Charlesa S. Peirce'a* (The Neglected Argument of C.S. Peirce)

- 22.04.2010: Jan Woleński, *Dlaczego nie można (?) udowodnić istnienia Boga?* (Why it is Impossible (?) to Prove God's Existence?)
- 21.10.2010: Zbigniew Wolak, *Jan Paweł II o relacjach między nauką i teologią* (John Paul II on the Relationship Between Science and Theology)
- 25.11.2010: Bogdan Dembiński, *Ewolucja platońskiej doktryny idei* (The Evolution of the Platonic Doctrine of Ideas)
- 16.12.2010: Robert Woźniak, *W drodze do teologii nauki* (Towards the Theology of Science)

III. Philosophy of Physics and Cosmology

Head

- Professor Marek Szydłowski (Catholic University of Lublin)

Members

- The Kraków/Lublin Team
  - Professor Marek Szydłowski (team coordinator, Catholic University of Lublin)
  - Professor Marek Biesiada (Silesian University)
  - Fr. Dr. Jacek Golbiak (Catholic University of Lublin)
  - Dr. Adam Krawiec (Jagiellonian University)
  - Dr. Monika Hereć (Catholic University of Lublin)
  - Fr. Dr. Paweł Tambor (Catholic University of Lublin)
  - Orest Hrycyna, M.A. (Catholic University of Lublin)
  - Aleksandra Kurek, M.A. (Jagiellonian University)
  - Jakub Mielczarek, M.A. (Jagiellonian University)
  - Łukasz Kukier, M.A. (UMCS Lublin)
- The Szczecin Team
  - Professor Mariusz P. Dąbrowski (team coordinator, Szczecin University)

- › Professor Janusz Garecki (Szczecin University)
- › Professor Jerzy Stelmach (Szczecin University)
- › Dr. Tomasz Denkiewicz (Szczecin University)
- › Adam Balcerzak, M.A. (Szczecin University)

Research Fields

- » axiology of modern cosmology
- » temporality of modern cosmology
- » philosophical assumptions in cosmology
- » a study of the boundaries of physics and cosmology
- » complex systems – conceptual foundations and philosophical aspects
- » cosmobiology
- » the beginning of the Universe in modern cosmology
- » the notion of multiverse in modern cosmology
- » Feynman's notion of quantum gravity

Recent activities

The current research of the Philosophy of Physics and Cosmology group focuses on four main subject areas. The first is the study and elaboration of the philosophical aspects of the notion of emergence as applied to the problems of physics and cosmology. The significance of such interdisciplinary studies in both science and philosophy is unquestionable. It may lead to deeper understanding of the still unclear conception of philosophical emergence as an explanatory tool in the context of scientific theories. The concept of emergence is studied in connection with mathematics as well. The second field of investigation is devoted to the study of the sociological aspects of modern science. The relevance of the ideas of L. Fleck in this context is of special interest. Thirdly, in physics an important role is played by the so-called non-empirical criteria of theory acceptance, such as simplicity, symmetry, beauty etc. The methodological and philosophical role of these criteria has been analyzed in the context of physics and cosmology. The last domain of the team's research is the study of complex systems of cosmological and physical origins. Especially, the notion of dynamic complexity methods and their application in cosmology for improving the evolutionary scenario is analyzed. This leads to analyzing profound questions in the philosophy of complexity.

The research results have been presented in the following papers:

- J. Pietrak, M. Szydłowski, P. Tambor, „Fraktale: konstrukcja czy emergencja? Część I. Fraktalne jednostki emergencji w klasycznym schemacie pojęciowym", *Zagadnienia Naukoznawstwa*, 183 (2010), p. 43-64.
- J. Pietrak, M. Szydłowski, P. Tambor, „Fraktale: konstrukcja czy emergencja? Część II. Emergencja fraktalna w podejściu quasi-empirycznym", *Zagadnienia Naukoznawstwa*, 184 (2010), p. 271-302

In the research project we form a conceptual framework that enables the characterisation of fractal structures from the point of view of philosophical concept of emergence. In the first part, we present the main ideas of the philosophy of emergence as well as an attempt at capturing emergent units in the process of fractals' generation. However, we stick to the classical understanding of the relation in question. In the second part, due to the demonstration of weaknesses of the classical accounts as insufficient in the specific context of mathematical structures, we show that the discourse about emergence in mathematics becomes meaningful and valid through the adaptation of quasiempirical approach towards some issues in mathematics, an approach grounded in the philosophy of formal sciences.

» M. Hereć, M. Szydłowski, P. Tambor, *Samoorganizujący się Wszechświat w różnych skalach — miejsce, gdzie nauka spotyka się z filozofią* (Transfer idei. Od ewolucji w biologii do ewolucji w astronomii i kosmologii, KUL, Lublin 18-19 listopada 2009).

We review a broad class of complex self-organizing systems existing in nature. In the context of these systems we point out the philosophical implications for the discussion of causality, holism vs. reductionism and emergence. We propose a philosophical notion of emergence with emergence investigated in science for the deeper and more precise formulation and understanding of this concept. We also stress the importance of the ontological question: why are self-organizing systems actually possible?

» M. Szydłowski, P. Tambor, *Kosmologia współczesna w schemacie pojęciowym kolektywu badawczego i stylu myślowego Ludwika Flecka* (Racjonalne kontra socjologiczne rekonstrukcje rozwoju wiedzy, UMCS, Lublin 1 Czerwca 2010).

In this paper we try to distinguish 'the styles of scientific thinking' in the specific sense of L. Fleck's philosophy of science as applied to the modern cosmology. We investigate two distinct styles: the factual and the theoretical one. We show how important it is to take into consideration the impact of the sociological factors on the scientific research.

» M. Szydłowski, P. Tambor, „Prostota modelu kosmologicznego a złożoność Wszechświata", *Roczniki Filozoficzne*, 58 (2010), p. 153-180.

In this paper we continue our investigation of Akaike simplicity criterion, which plays a crucial role in E. Sober's philosophy of science. We consider different models of accelerating Universe which describe the current evolution of the Universe. We demonstrate that the generalized Akaike criterion (Bozdogen criterion) identifies a very simple standard cosmological model called LCDM (Lambda Cold Dark Matter Model), whereas the standard Akaike criterion doesn't provide such a unique identification. We demonstrate that the analysis of cosmological models may be useful in searching for more adequate criterion of the simplicity in the context of the philosophy of science.

» D. Ciszewska, M. Szydłowski, „Piękno jako przykład poza-empirycznego kryterium w akceptacji teorii naukowej", *Zagadnienia Filozoficzne w Nauce*, XLVII (2010).

It can be said that beauty and science do not share a common ground yet it is not necessarily true. The place of beauty in science is between two branches of knowledge: aesthetics and physics. The purpose of the paper is to show that they are connected, permeate each other and operate together to yield scientific results. The scientific theories and the aesthetic theories display some similarities, not only at the construction stage, but also in their use. The paper examines two aesthetic criteria — symmetry and simplicity — which are very valuable tools of choosing the scientific theory. The most important aspect of 'beauty in science' is the fact that aesthetic criteria influence the choice of scientific theory.

» T. Stachowiak, M. Szydłowski, "A differential algorithm for the Lyapunov spectrum", [arXiv:1008.3368v2].

We present a new algorithm for computing the Lyapunov exponents spectrum based on a matrix differential equation. This approach belongs to the so-called continuous type, where the rate of expansion of perturbations is obtained for all times, and the exponents are reached as the limit at infinity. It does not involve exponentially divergent quantities so there is no need for the rescaling or realigning of the solution. We show the method's advantages and drawbacks using the example of a particle moving between two contracting walls.

IV. Mathematical Structures of the Universe

Head

» Professor Andrzej Woszczyna (Jagiellonian University)

Members

» Fr. Professor Michał Heller (John Paul II Pontifical University)
» Professor Zdzisław Golda (Jagiellonian University)
» Dr. Jacek Gruszczak (Pedagogical University, Kraków)
Professor Wiesław Sasin (Warsaw Technical University)
» Professor Andrzej Sitarz (Jagiellonian University
» Professor Leszek Sokołowski (Jagiellonian University
» Dr. Zdzisław Odrzygóźdź (Warsaw Technical University
» Dr. Leszek Pysiak (Warsaw Technical University)
» Dr. Wojciech Czaja (Jagiellonian University)

Research Fields

» interpretational issues in the applications of noncommutative geometry to physics
» mathematical formulations of gravitational physics
» the theory of structure formation in the Universe
» computer algebra systems with application to general relativity

Recent activities

The publications of the members of the research team in 2010 include:

Books

- M. Heller, *To Catch the Transient Moment*, second edition, Znak, Kraków 2010 (in Polish);
- M. Heller, *The Sense of Life and the Sense of the Universe. Studies in Contemporary Theology*, translated by Aeddan Shaw, Karolina Czerska-Shaw, Copernicus Center Press, Kraków 2010.

Papers

- M. Heller, "A Commutative Friedman Cosmological Model", *Annalen der Physik*, 19 (2010), p. 196-201;
- M. Heller, "A Self-Contained Universe?" [in:] *The Trinity and an Entangled World. Relationality in Physical Sciences and Theology*, J. Polkinghorne (ed.), Earmans Publishing Company, Grand Rapids. Michigan, Cambridge UK, 2010, pp. 41-54;
- A. Sitarz, A. Zajac, 2010, "Spectral action for scalar perturbations of Dirac operators" [arXiv:1012.3860];
- L. Dabrowski, A. Sitarz, 2010, "Noncommutative circle bundles and new Dirac operators", [arXiv:1012.3055];
- P. Olczykowski, A. Sitarz, 2010, "On spectral action over Bieberbach manifolds", [arXiv:1012.0136];
- B. Iochum, T. Masson, T. Schacker, A. Sitarz, 2010, "Compact κ-deformation and spectral triples", [arXiv:1004.4190];
- J. Gruszczak, 2010, "Smooth Beginning of the Universe", [arXiv:1011.3824].

Workshops on Noncommutative Geometry

On February 9-12, 2011, the group met at the Diocesan Seminary in Tarnów to continue its work on noncommutative geometry. In the event participated: Michał Ekstein, Jacek Gruszczak, Michał Heller, Tomasz Miller, Zdzisław Odrzygóźdź, Leszek Pysiak, Wiesław Sasin. We continued our work on the semidirect product of groupoids and, in particular, on a crossed product given by the action of the groupoid on the groupoid algebra. We also studied the regular representation of such a crossed product on a bundle of Hilbert spaces. All calculations and proofs have been successfully completed, and the work is ready to be edited.

We also initiated a new topic — the definition and properties of a sheave of noncommutative algebras. In particular, we have constructed the sheaf of von Neumann algebras. Moreover, Z. Odzrzygóźdź presented his work on the Kaluza-Klein model in terms of the groupoid theory, and J. Gruszczak presented his work entitled "Smooth Beginning of the Universe" (arXiv:1011.3824 [gr-qc]).

There were also organized other workshops: 8-11 July 2000, Pasierbiec; 1-5 October 2010, Warsaw; 24-27 February 2011, Warsaw; 23-25 April 2011, Warsaw.

Lectures

Michael Heller delivered a series of lectures, *Comoslogy and Creation*, on 22-25 March 2010, at Universidad Internacional Menendez Pelayo, Tenerife, Canary Islands:

Lecture 1: The evolution and the structure of the Universe
A brief history of 20th century cosmology. From the static world to the expanding universe. Geometry of the Universe. Exploring the history of the Universe. Discovery of microwave cosmic

background radiation and its cosmological significance. Dark matter and dark energy. Some open questions.

Lecture 2: The origin of the Universe
Big Bang and the initial singularity problem. Various attempts to neutralize singularities. Hawking and Penrose singularity theorems. Malicious singularities. The search for the final physical theory. Is the Universe singular?

Lecture 3: Creation of the Universe
A brief history of the creation concept. Time and eternity: creation in time or creation with time? The concept of creation and the concept of beginning. Creation of the Universe and modern cosmology. Did God create the Universe in the Big Bang? "God of the gaps" strategy.

V. Philosophy and History of Physics

Head

- Fr. Dr. Wojciech P. Grygiel (John Paul II Pontifical University)

Members

- Fr. Dr. Robert Janusz (Ignatianum, Kraków)
- Fr. Dr. Tadeusz Pabian (John Paul II Pontifical University)
- Dr. Andrzej Koleżyński (John Paul II Pontifical University / University of Science and Technology, Kraków)
- Fr. Dr. Łukasz Mścisławski OP (John Paul II Pontifical University)

Research Fields

- the history of 20th Century physics
- the philosophical interpretation of physical theories
- the evolution of concepts in physics

Recent Activities

The activity of the research group of Philosophy and History of Physics focuses mainly on the studies of the ontological status of physical theories and their relation to the physical reality that they purport to describe. There seems to be no general agreement among contemporary physicists in this regard. The prevailing attitude, however, relies on the concept put forward by W. V. O. Quine whereby the existence of physical entities is in a broadened sense warranted by the formalism of the theory. This approach permits the exercise of a philosophical inquiry without commitment to any *a priori* assumed ontology of the physical reality. In conjunction to these efforts, a new interest has emerged within the group that prompted studies in the area of the philosophy of mind in cooperation with the Copernicus Center Neuroscience research team. In particular, philosophy of mind exercised in the context of cognitive sciences opens up possibilities to investigate classical philosophical problems of the human mind (e.g., the mind-body problem) in the light of the achievements of contemporary sciences. Thus, a new branch of the so-called philosophy in science finds its good foundations. A considerable effort of the group's activity was also devoted to the study of the philosophical implications of Bell's theorem and the completion of the project entitled 'Experimental Metaphysics'.

Selected publications

» Wojciech P. Grygiel, „Czy czas jeszcze płynie w fizyce? Ontologia czasu a współczesne teorie fizyczne", *Logos i Ethos*, 28 (2010), pp. 107-120.

» Wojciech P. Grygiel, „Matematycy o matematycznym platonizmie. Zapis ciekawej dyskusji„ *Logos i Ethos*, 29 (2010), pp. 7-26.

- Wojciech P. Grygiel, „Dlaczego kwantowanie grawitacji może interesować filozofa?" *Postępy fizyki*, 61 (2010), pp. 211-215.
- Wojciech P. Grygiel, „Jak uniesprzecznić sprzeczność umysłu?„ *Zagadnienia filozoficzne w nauce*, 47 (2010), pp. 70-88.
- Wojciech P. Grygiel, „Teoria superstrun i Lee Smolina kłopoty z fizyką", *Filozofia Nauki*, 71 (2010), pp. 141-152.
- Wojciech P. Grygiel, "The quantum world – real but not measurable" [in:] Mariola Kuszyk-Bytniewska, Andrzej Łukasik (ed.), *Filozofia przyrody współcześnie*, Kraków: Universitas 2010, pp. 131-150.
- Tadeusz Pabjan, „Krótka (pre)historia argumentu EPR", *Zagadnienia Filozoficzne w Nauce*, 47 (2010), pp. 54-69.
- Tadeusz Pabjan, „Problem upływającego czasu", *Postępy Fizyki*, 61/2 (2010), pp. 77-79.
- Tadeusz Pabian, „George'a FitzGeralda hipoteza «kontrakcji»", *Kwartalnik Historii Nauki i Techniki*, 55/2 (2010), pp. 169-185.

VI. History of Mathematics: People-Ideas-Philosphical Aspects

Head

- Professor Wiesław Wójcik (Institute of the History of Science, PAN)

Members

- Profesor Roman Duda (Wrocław University)
- Professor Grażyna Rosińska (Institute of the History of Science, PAN)
- Professor Krzysztof Maślanka (Institute of the History of Science, PAN)

- Professor Zbigniew Król (Institute of Philosophy and Sociology, PAN)
- Professor Jerzy Dadaczyński (John Paul II Pontifical University)
- Dr. Zdzisław Pogoda (Jagiellonian University)
- Dr. Stanisław Domoradzki (University of Rzeszow)
- Dr. Zofia Pawlikowska - Brożek
- Dr. Gabriela Teresa Besler (Silesian University)
- Dr. Andrzej Brzoza (Silesian Technical University)
- Dr. Paweł Turkowski (University of Rzeszów)

Research Fields

- history of Polish mathematics
- conceptions of the unity of mathematics
- philosophical foundations of mathematics
- the changeability of the notion of 'mathematics'
- the evolution and meaning of the mathematical 'basic concepts'
- ancient, modern and contemporary mathematics: differences and their philosophical sources

Recent activities

In the academic year 2009/2010 six seminars of the research team took place. The following papers were presented:

- „Metodologiczny status historii matematyki. Część I" (Methodological status of the history of mathematics. Part I), Professor Wiesław Wójcik, Kraków, Instytut Matematyki i Informatyki UJ, Łojasiewicza Str. 6, 16 October 2009.
- „Dedekind, Cantor, Kronecker", Professor Jerzy Mioduszewski, Katowice, Wydział Nauk Społecznych Uniwersytetu Śląskiego, Bankowa Str. 11, 20 November 2009.

- „Metodologiczny status historii matematyki. Część II" (Methodological status of the history of mathematics), Professor Wiesław Wójcik, Kraków, Instytut Matematyki i Informatyki UJ, Łojasiewicza Str. 6, 18 December 2009.
- „Historia matematyki na Kongresie Historii Nauki w Budapeszcie (wrzesień 2009)" (History of mathematics at the Congress of the History of Science in Budapest, September 2009), Dr. Zdzisław Pogoda and Dr. Stanisław Domoradzki, Kraków, Instytut Matematyki i Informatyki UJ, Łojasiewicza Str. 6, 26 February 2010.
- „Teoria idei w kontekście platońskiej filozofii matematyki" (Theory of ideas in the context of the Platonic philosophy of mathematics), Professor Bogdan Dembiński, Katowice, Wydział Nauk Społecznych Uniwersytetu Śląskiego, Bankowa Str. 11, 19 March 2010.
- „Komputery w matematyce. Dowody wspomagane komputerowo" (Computers in mathematics. Computer-assisted proofs), Professor K. Maślanka, Kraków, Instytut Matematyki i Informatyki UJ, Łojasiewicza Str. 6, 17 June 2010.

VII. Neurobiology

Head

- Professor Jerzy Vetulani (Polish Academy of Science)

Members

- Professor Dominika Dudek (Collegium Medicum of the Jagiellonian University)
- Professor Janusz Rybakowski (Medical University, Poznań)

Research Fields

- Research connected with experimental work on the functioning of the human brain as well as the question of its interpretation and methodological connection with neurobiology.

Recent Activities

- An informal discussion group, *Trapez*, has been established. The meetings of the group are held once a month. The participants are: Professor J. Vetulani, Professor I. Nalepa, Professor J. Silberring, B. Kaszkur, K. Bielawski, Fr. G. Babiarz, rev. Dr. W. Grygiel, Dr. M. Siwek, Professor D. Dudek. The discussions are devoted to the problems of neurobiology in its relations with other disciplines, such as the humanities, and the neuroscientific understanding of human behaviour (e.g., aggression, morality, etc.).
- During the Congress of the Polish Psychiatric Society in Poznań (23-26.06.2010) there was organized a special session under the patronage of the Copernicus Center, *Science in Search of the Soul*. The lectures were delivered by: Professor Jerzy Vetulani, Professor Janusz Rybakowski, Professor Jerzy Aleksandrowicz, Professor Dominika Dudek.
- During *The Brain Week* in Kraków the members of the group delivered the following lectures: Professor Jerzy Vetulani, "Can the scalpel help to heal the soul? The Perspective of Psychiatry from a Biological Point of View"; Professor Dominika Dudek, "Psychotherapy and Neuroscience" (both in Polish). The lectures were subsequently published in the *Wszechświat* monthly.

- Professor Jerzy Vetulani published a book monograph *Mózg – fascynacje, problem, tajemnice* (Brain – Fascinations, Problems, Mysteries), Wydawnictwo Homini, 2010. The goal of the papers included in the book is to popularize neuroscience and try to show that the 'big questions' posed by philosophers, theologians, ethicians and psychologists may be addressed by neuroscience.
- Professor Jerzy Vetulani delivered a number of popular lectures.
- Within the cooperation between Klinika Psychiatrii Dorosłych in Poznań, Katedra Psychiatrii in Wrocław and Klinika Psychiatrii Dorosłych in Krakow, a research project TRES DEP was carried out. The project's goal is to assess the risk of bi-polar disorder in patients with treatment-resistant depression. The directors of the project, Professors Janusz Rybakowski and Dominika Dudek are the members of the *Neuroscience* research team. Some of the papers published within the project were affiliated with the Copernicus Center: A. Kiejna, T. Pawłowski, D. Dudek, D. Łojko, M. Siwek, R. Roczeń, J. K. Rybakowski, "The utility of Mood Disorder Questionnaire for the detection of bipolar diathesis in treatment-resistant depression", *Journal of Affective Disorders*, 124 (2010), pp. 270-274; D. Dudek, J. K. Rybakowski, M. Siwek, T. Pawłowski, D. Łojko, R. Roczeń, A. Kiejna, "Risk factors of treatment resistance in major depression", *Journal of Affective Disorders*, 126 (2010), pp. 268-271.

VIII. Methodology and Philosophy of Science

Head

- Professor Wojciech Załuski (Jagiellonian University)

Members

- Professor Bartosz Brożek (Jagiellonian University)
- Fr. Dr. Zbigniew Liana (John Paul II Pontifical University)
- Fr. Professor Adam Olszewski (John Paul II Pontifical University)
- Professor Anna Brożek (Warsaw University)

Research Fields

- classical philosophy of science (Vienna Circle, Popper, Kuhn, Lakatos)
- methodology of biosciences
- methods in social sciences

Recent activities

The current research includes of the group is concentrated on:

- the notion of the mathematical subject in the philosophy of mathematics of Kant, Hilbert, Frege and Brouwer;
- the logic of non-foundationism;
- the philosophy of K.R. Popper;
- the applications of game theory in philosophy;
- the philosophy of social sciences.

There were published:

- a monograph by Bartosz Brożek, *The Double Truth Controversy. An Analytical Essay*, Copernicus Center Press, Kraków 2010;
- several papers in Polish and English: Brożek, B. and A. Olszewski (2010): "Kilka uwag o kryterium Quine'a" (Some Remarks on Quine's Criterion), *Filozofia Nauki*; A. Olszewski, "Intuicjonizm vs Platonizm. Na przykładzie

lematu Koeniga" (Intuitionism and Platonism. On the Example of Koenig's Lemma) [in:] I. Bondecka-Krzykowska, J. Pogonowski (eds.), *Światy matematyki. Tworzenie czy odkrywanie?*, Poznań: Wydawnictwo Naukowe UAM 2010, pp. 261-274; A. Olszewski, „Uwagi o Rajmundzie Lullusie, jego dziele i logice treści" (Remarks on R. Lullus, his work and the material logic), *Logos i Ethos*, 28 (2010), p. 89-106.

IX. Analytical Metaphysics

Head

- Professor Tomasz Placek (Jagiellonian University)

Members

- Professor Andrzej Wroński (Jagiellonian University
- Professor Tomasz Bigaj (Warsaw University)
- Dr. Jerzy Gołosz (Jagiellonian University)
- Dr. Leszek Wroński (Jagiellonian University)
- Jacek Wawer, MA (Jagiellonian University)

Research Fields

- causality theories in the classical and probabilistic versions
- determinism in nature – determinism of scientific theories
- possible-worlds structures in connection to space-time structures
- metaphysical implications of some physical results such as Bell's theorems

Recent activities

The year 2010 brought a new impetus to the functioning of the group as it received a MNiSW grant. Technically, the grant is to support Polish philosophers participating in a programme of the European Foundation of Science called „Philosophy of Science: A European Perspective". Tomasz Placek is a member of the programme's steering committee and represents Poland in it. The grant has permitted the younger members of the group to work full time on their research. It has also allowed members of the group to participate in international conferences or workshops, where they presented their results, or exchanged ideas with other researchers.

The core of the group consists of three researchers, Professor Tomasz Placek, Dr. Leszek Wroński and Jacek Wawer, MA. Additionally, there are about five PhD students or advanced undergraduate students who regularly come to the group's meetings.

In 2010 the group held regular Friday meetings, of three kinds. First, there has been a reading group that discussed main papers in the project of branching space-times (and did proofs presented in these papers). The aim of the reading group is to acquire mathematical and philosophical sophistication needed to work in this field. The second team attempts an analysis of Bell-type theorems in a framework that is both stochastic and modal. The novelty of this project lies exactly in the combination of stochasticity and modality. I estimate the project will come to fruition in summer 2011. Finally, the third team investigates a specific project of diagnosing Bell's theorem, initiated by Hofer-Szabo et al. This work is almost complete, as T. Placek and L. Wroński have just finished a paper which proves that the mentioned project is mathematically incorrect.

Apart from the collaborative work, the group members individually pursue their own projects. Jacek Wawer has just submitted to *Synthese* a paper (jointly with A. Malpass) on Ockhamist's tense logics, defending a notion that there is a distinguished future history (i.e., they defend the so-called Thin Red

Line doctrine). Leszek Wroński continues his investigations of common causes and related notions, which were the topic of his doctoral dissertation. Tomasz Placek has recently published his construction of semantics for modal languages that uses possible continuations rather than possible worlds. He also developed an argument showing counterintuitive consequences of analyzing the present in modal terms.

The highlight of the group's activity in 2010 was a workshop "Causes and Tenses: Formal Perspectives", which the group organized in September 10-11. The workshop was sponsored jointly by the Copernicus Center and Department of Philosophy of the Jagiellonian University, and was generally assessed as very successful. The workshop's underlying idea was to bring together researchers doing probabilistic causation (analyzed formally) with researchers of tenses, analyzed from either a logical or metaphysical perspective. The emphasis was on small data structures for analyzing causes and indeterminism, common causation and causal completability, and new techniques for the problem of future contingents. The event gathered distinguished speakers: Nuel Belnap (Pittsburgh), Fabrice Correia (Geneva), Thomas Müller (Utrecht) and Miklós Rédei (London). The journal *Erkenntnis* has agreed to put together a special issue comprising papers read at this workshop. This issue will be edited by T. Placek, J. Wawer and L. Wroński.

In 2010 the group members published the following papers that funded or partly funded through the MNiSW grant administered by the Copernicus Center:

» T. Placek: "Possibilities without possible worlds / histories", *Journal of Philosophical Logic* (2010), DOI: 10.1007/s10992-010-9159-x;

- T. Placek and N. Belnap: "Indeterminism is a modal notion: branching spacetimes and Earman's pruning", *Synthese* (2010), DOI: 10.1007/s11229-010-9846-8;
- T. Placek: "A locus for 'now'", [in:] D. Dieks, W. J. Gonzalez, S. Hartmann, T. Uebel and M. Weber (eds.), *Explanation, Prediction, and Confirmation: New Trends and Old Ones Reconsidered*, Dordrecht: Springer, pp. 399-41;

In 2010 the members of the group participated in the following conferences and workshops:

- BIRTHA Postgraduate Conference on Branching Time and Indeterminacy, Bristol August 2010, participated: T. Placek, L. Wroński and J. Wawer. T. Placek and J. Wawer read papers, T. Placek was an invited speaker;
- ESF workshop „Points of Contact between the Philosophy of Physics and the Philosophy of Biology: Probability, Laws and Natural Kinds", London School of Economics, London, UK, December 13-15, participated: T. Placek, L. Wroński.

X. Polish Philosohy of Nature in the First Half of the 20th Century

Head

- Fr. Professor Janusz Mączka (John Paul II Pontifical University)

Members

- Dr. Paweł Polak (John Paul II Pontifical University)
- Dr. Andrzej Koleżyński (John Paul II Pontifical University)
- Anna Madej, M.A. (John Paul II Pontifical University)
- Małgorzata Stawarz, M.A. (John Paul II Pontifical University)

Research Fields

- basic ideas of the Polish philosophy of science in the first half of the 20th century
- the main representatives of the Polish philosophy of science in the first half of the 20th century
- the peculiarity of the Polish philosophy of science against the backdrop of international philosophy of science
- preparation of pre-war manuscripts concerning the philosophy of science

Recent activities

In 2010, there were held four meetings of the research group:

- 12 January 2010: Dr. Paweł Polak presented his paper "Perspektywy badań nad historią polskiej filozofii przyrody (The Perspectives on the Research on the History of the Polish Philosophy of Nature). He concentrated on the following two issues: 1) the reception of the evolutionary theory in Poland; and 2) the debate concerning Darwinism at the break of 19th and 20th centuries.
- 19 March 2010: Dr. Paweł Polak presented the project of developing a Virtual Library and the Archive of the Copernicus Center. The goal of the project would be to collect and make accessible online various resources connected to the history of science and philosophy in Poland. Anna Madej, MA, read her paper on *Problematyka celowości w przyrodzie ożywionej w ujęciu J. Metallmanna* (The Problem of Teleology in Nature According to J. Metallmann), based on an unpublished manuscript of Metallmann. The presentation was followed by a discussion which highlighted

the peculiar character of the principle of teleology in Metallmann's holistic philosophy of nature.

- 21 June 2010: the meeting was devoted to the review of the work of the research group.
- 1 December 2010: the meeting was devoted to the project of publishing a number of unpublished papers by Marian Smoluchowski.

XI. Biological Foundations of Law and Ethics

Head

- Professor Bartosz Brożek (Jagiellonian University)

Members

- Professor Jerzy Stelmach (Jagiellonian University)
- Professor Wojciech Załuski (Jagiellonian University)
- Marcin Gorazda, M.A. (John Paul II Pontifical University)
- Łukasz Kurek, M.A. (Jagiellonian University)
- Radosław Zyzik, M.A. (Jagiellonian University)
- Jakub Kabza, M.A. (Jagiellonian University)
- Aeddan Shaw, M.A. (Tischner European University)

Research Fields

- ethics and neurosciences
- law and neurosciences
- the concept of normativity
- conceptual schemes in law and ethics
- the evolutionary model of ethics and law
- evolutionary theory in social sciences
- the methodology of social sciences

Recent activities

The group meets once a week. The meetings of the group are documented at the *Biolawgy* blog: www.biolawgy.wordpress.com. The main directions of current research include:

» naturalism and anti-naturalism in philosophy and in the normative sciences;
» neuroscience and the law: the foundations of private and criminal law against the background of neuroscientific theories;
» ontology of rules and the problem of normativity — the insights from the evolutionary theory and neuroscience.

In 2010, the members of the group published a book monograph:

» J. Stelmach, B. Brożek, W. Załuski, M. Soniewicka, *Paradoksy bioetyki prawniczej* (Paradoxes of Legal Bioethics), Wolters Kluwer Polska, Warszawa 2010.

In collaboration with the Department for the Philosophy of Law and Legal Ethics of the Jagiellonian University two collections of essays were also published:

» J. Stelmach, M. Soniewicka, W. Załuski (eds.), *Studies in the Philosophy of Law IV: Legal Philosophy and the Challenges of Biosciences*, Jagiellonian University Press, Kraków 2010;
» J. Stelmach, M. Soniewicka, B. Brożek (eds.), *Studies in the Philosophy of Law V: Law and Biology*, Jagiellonian University Press, Kraków 2010.

The members of the team published the following papers pertaining to the biological foundations of law and ethics:

» B. Brożek, R. Zyzik, „Reguły prawne z perspektywy *Dociekań filozoficznych*" (Legal Rules from the Perspective

of *Philosophical Investigations*), *Ruch Prawniczy, Ekonomiczny i Socjologiczny*, LXXII/2 (2010), p. 113-132.

» B. Brożek, „Normatywność prawa. Szkic teorii", *Logos i Ethos* 28 (2010), p. 29-66.

» B. Brożek, "The Notion of the Person in Bioethical Debates", *Studies in the Philosophy of Law IV*, Jagiellonian University Press, Kraków 2010, p. 85-96.

» B. Brożek, "Some Remarks on the Naturalization of Law", *Studies in the Philosophy of Law V*, Jagiellonian University Press, Kraków 2010, p. 73-82.

» W. Załuski, "Genetic Enhancement and Human Autonomy", [in:] M.Fernández-Barrera et al. (ed.) *Law and Technology: Looking into the Future - Selected Essays*, European Press Academic Publishing, Florence, Italy 2010, p. 255-270.

» W. Załuski, "Human Nature After Darwin", [in:] J. Stelmach, M. Soniewicka, W. Załuski (eds.), *Studies in the Philosophy of Law IV: Legal Philosophy and the Challenges of Biosciences*, Jagiellonian University Press, Kraków 2010, pp. 77-84.

» A. Giza, W. Załuski, "Ethical Problems of Hematopoietic Stem Cell Transplantation", [in:] J. Stelmach, M. Soniewicka, W. Załuski (eds.), *Studies in the Philosophy of Law IV: Legal Philosophy and the Challenges of Biosciences*, Jagiellonian University Press, Kraków 2010, pp. 165-171.

» J. Stelmach, "Six Causes of the Dispute Over Legal Bioethics", [in:] J. Stelmach, B. Brożek, M. Soniewicka (eds.), *Studies in the Philosophy of Law V: Law and Biology*, Jagiellonian University Press, Kraków 2010, pp. 9-18.

» Radosław Zyzik, "Neurolaw. A New Paradigm in Legal Philosophy?", in: J. Stelmach, B. Brożek, M. Soniewicka (eds.), Studies in the Philosophy of Law V: Law and Biology, Jagiellonian University Press: Kraków 2010, pp. 125-134.

Three PhD dissertations were also submitted:

- Łukasz Kurek, *Wina w prawie karnym a problem wolnej woli w świetle neurofilozofii* (Guilt in Criminal Law And the Problem of Free Will in the Light of Neurophilosophy)
- Radosław Zyzik, *Antropologiczne założenia teorii oświadczeń woli z perspektywy neurokognitywnej* (The Anthropological Assumptions of the Theory of The Meeting of the Minds from a Neurocognitive Perspective)
- Marcin Gorazada, *Obrona antynaturalizmu* (In Defence of Anti-naturalism)

Publications

Copernicus Center Reports #1

In 2010 the first volume of the *Copernicus Center Reports* was published. The volume consists of two parts. The first is *Annual Report 2009*, a record of research, publishing, education and conference activities of the Copernicus Center in 2009. The second, *Essays*, comprises three papers: M. Heller, *The Struggle for Meaning*, W. Załuski, *Human Nature after Darwin*, and A. Pelczar, *Stanisław Zaremba*.

Copernicus Center Press

In 2010 the Copernicus Center Press was inaugurated. In its inaugural year, two books were published:

» Bartosz Brożek, *The Double Truth Controversy. An Analytical Essay*, Kraków 2010, ISBN 978-83-62259-01-4.

The principal source of the double truth controversy is the condemnation of 1277 issued by the bishop of Paris, Stephan Tempier. In the introduction to the document, Tempier condemns those philosophers who claim that there are things true according to the Catholic faith but false on the basis of natural reason. There is little doubt that Tempier thought of Siger of Brabant and Boethius of Dacia. There are several problems involved here, the first being whether anyone has actually ever advocated double truth. The second follows on the heels of the first since, if so, what does the theory of *duplex veritas* amount to? Finally, is it rational to uphold a view that embraces double truth? This book attempts to answer these questions via an exegesis of historical texts and by the application of some modern logical techniques to medieval theories. The resulting picture is that of complex and interesting accounts of the relationship between faith and reason, enabling an in-depth reflection on the contemporary discussions of the issue.

» Michael Heller, *The Sense of Life and the Sense of the Universe*, Kraków 2010, ISBN 978-83-62259-02-1.

Questions concerning the sense of man and the sense of the universe are closely related. In fact, they constitute one Big Question. Man is genetically connected to the universe and the origins of man have their roots in the history of the universe. If the universe has sense then it most probably embraces man since he is a part or, even more, an element of the structure of the universe. Would it be possible for man to have a senseful existence in a senseless universe?

Zagadnienia Filozoficzne w Nauce

Copernicus Center, in cooperation with Center for Interdisciplinary Studies (OBI) and Biblos Publishing House, continues to publish *Zagadnienia Filozoficzne w Nauce*. In 2010 the 46th volume of this periodical was published.

Studia Copernicana

Studia Copernicana is a series of book monographs and collections of essays published by the Institute for the History of Science of the Polish Academy of Sciences in cooperation with Copernicus Center. In 2010 the following was published: Jerzy Dobrzycki, *Selected Papers on Medieval and Renaissance Astronomy*, ed. by J. Włodarczyk and R.L. Kremer.

Education (Copernicus College)

Science and the Big Questions

In 2009/2010 academic year, the Copernicus Center, in cooperation with the University of Information Technology and Management in Rzeszów and Tischner European Unviersity in Kraków organized 14 open lectures (in Polish) within the series *Science and the Big Questions*. The lectures were transmitted online at the Center's webpage. The list of delivered lectures is as follows:

1. Z. Liana, ,,Filozofia nauki #3" (Philosophy of science #3)
2. J. Vetulani, ,,Neurobiologia dokonywania wyboru moralnego" (Neurobiology of moral choice)
3. M. Szafrański, „Neuroscience a filozofia" (Neuroscience and philosophy)
4. D. Dudek, „Psychoterapia w świetle neuroscience" (Psychotherapy in light of neuroscience)
5. W. Grygiel, „Filozofujący fizycy" (Philosophising physicists)
6. T. Pabian, „Mechanika kwantowa a granice czasu i przestrzeni" (Quantum mechanics and the limits of time and space)
7. M. Heller, „Czy wszechświat wyjaśnia się wszechświatem?" (Can we explain universe with the universe itself?)
8. S. Wszołek, „Czy nauka eliminuje metafizykę?" (Does science eliminate metaphysics?)

9. A. Koleżyński, „Emergencja - nowa moda czy konieczność?" (Emergence — new fashion or necessity?)
10. M. Heller, „Nauka a wiara" (Science and faith)
11. P. Koteja, ,,Eksperymentalne badanie darwinowskiej teorii ewolucji" (Experimental studies of the Darwinian evolutionary theory)
12. P. Polak, „Recepcja teorii ewolucji" (Reception of the evolutionary theory)
13. J. Mączka, „Nauka polska a wielkie pytania (wybrane aspekty)" (Polish science and the big questions)
14. T. Placek, ,,Nierówności Bella: metafizyka eksperymentalna?" (Bell's inequalities: experimental metaphysics?)

Science and Religion

In 2010/2011 academic year, the Copernicus Center, in cooperation with the University of Information Technology and Management in Rzeszów and Tischner European University in Kraków organized 21 open lectures (in Polish), within the series *Science and Religion*. The lectures were transmitted online at the Center's webpage. The list of lectures is as follows:

1. Michał Heller: „Pierwsze konflikty miedzy rozumem i religią" (First conflicts between science and religion)
2. Teresa Obolevitch: „Rozum a wiara w pierwszych wiekach chrześcijaństwa" (Reason and faith in the early christianity)
3. Zbigniew Liana: „Wczesnośredniowieczne próby zmagania się z problemem rozumu w obrębie wiary" (Early medieval attempts at the reconciliation of faith and reason)

4. Bartosz Brożek: „Wielka Scholastyka. Rozum w opozycji do wiary" (The great scholasticism: faith in opposition to reason)
5. Włodzimierz Skoczny: „Rozum a wiara w renesansie" (Faith and reason in renaissance)
6. Janusz Mączka: „Czternastowieczne kontrowersje wokół relacji nauki i wiary" (Faith and reason in the 14^{th} century)
7. Zbigniew Liana: „Okultyzm a nauka przed Oświeceniem" (Occultism and science before Enlightement)
8. Janusz Mączka: „Kopernika problemy z Inkwizycją" (Copernicus and the Inquisition)
9. Włodzimierz Skoczny: „Sprawa Galileusza" (The case of Galileo)
10. Wojciech Grygiel: „Powstanie nauki a teologia" (The emergence of science and theology)
11. Dyskusja na temat: Rozum a wiara — konflikt czy współistnienie? (Panel discussion: Faith and reason: conflict or coexistence?)
12. Tadeusz Pabjan: „Nauka a początki nowożytnego materializmu" (Science and the beginning of modern materialism)
13. Tadeusz Pabjan: „Fizyko-teologia" (Physico-theology)
14. Bartosz Brożek: „Nauka i wiara. Aspekty metodologiczne" (Science and faith: methodological aspects)
15. Stanisław Wszołek: „Racjonalność wiary" (The rationality of faith)

16. Teresa Obolevich: „Recepcja ewolucjonizmu w kulturze rosyjskiej" (The reception of evolutionary theory in the Russian culture)
17. Włodzimierz Skoczny: „Nauki przyrodnicze w poglądach teologów: J.H. Newmana i S. Wilbeforce'a" (Science in the thought of theologians: J.H. Newman and S. Wilbeforce)
18. Wojciech Grygiel: „Relacja 'nauka — religia' w myśli Wernera Heisenberga" (Science and religion in the thought of Werner Heisengerg)
19. Jerzy Vetulani: „Neurobiologia i religia" (Neuroscience and religion)
20. Andrzej Koleżyński: „Nauka i humanizm: wpływ naukowego obrazu świata na rozwój duchowy człowieka w myśli Erwina Schrödingera" (Science and Humanism: the influence of the scientific worldview on the spiritual development of man according to Erwin Schrödinger)
21. Dyskusja panelowa: Nauka i Religia — przestrzeń dialogu (Panel discussion: Science and religion — the space of dialogue)

Conferences

I. *Copernicus Center Colloquium* #2 (Kraków, 16 January 2010)

During the second Copernicus Center Colloquium, the following papers were presented:

1. Adam Olszewski, ,,Uwagi filozoficzne o twierdzeniu o wachlarzu" (Philosophical Remarks on the Fan Theorem).
2. Wojciech Grygiel, ,,Stephena Hawkinga ontologia teorii fizycznych" (Stephen Hawking's Ontology of Physical Theories).
3. Paweł Polak, ,,Perspektywy badań nad historią polskiej filozofii przyrody" (Perspectives for Research on the History of the Polish Philosophy of Nature).

A report of the activities of the Copernicus Center was also presented.

II. *The Nicholas Copernicus grave mystery. A dialogue of experts* (Kraków, 22—23 February 2010)

The conference was co-organized by the European Society for the History of Science, the History of Science Commission of the Polish Institute of Arts and Sciences, the Philosophy of Natural Sciences Commission of the Polish Institute of Arts and Sciences, History of Science Institute of the Polish Academy of Science, Tischner European Unviersity in Kraków, and the Copernicus Center.

The goal of the conference was to discuss the controversies related with the discovery of the Nicolaus Copernicus grave and the identification of his remains. The participants of the conference were the scientists both partaking in the search, and criticising this undertaking.

During the conference, the following papers were presented (in Polish, or, when indicated, in English):

1. Jerzy Sikorski „Miejsce pochówku Mikołaja Kopernika w katedrze fromborskiej w świetle współczesnej praktyki" (The location of Nicholas Copernicus' burial in the light of the contemporary practice).
2. Krzysztof Mikulski, Joanna Jendrzejewska, and Anna Stachowska „Przodkowie i najbliżsi krewni Mikołaja Kopernika oraz ich żeńskie potomstwo" (Ancestors and close relatives of Nicholas Copernicus and their female descendants).
3. Tomasz Węcławowicz „Krypta biskupia we fromborskiej archikatedrze. Pytania badawcze" (The bishop's crypt in the Frombork Archcathedral. Research questions)
4. Wojciech Branicki, Tomasz Kupiec, „Analiza markerów DNA jądrowego w szczątkach z grobu 13/05" (The analysis of nuclear DNA markers in the remains from 13/05 grave).
5. Tomasz Kupiec, Wojciech Branicki, „Zastosowanie analizy mtDNA do identyfikacji szczątków ludzkich z grobu 13/05" (The application of the mtDNA analysis to the identification of human remains from 13/05 grave).
6. Arkadiusz Sołtysiak, „Myślenie magiczne w interpretacji archeologicznej. Przykłady, próba klasyfikacji, perspektywy" (Magical thinking in an archaeological interpretation. Examples, an attempt of classification, perspectives).

7. Jarosław Bednarek „Pułapka typologii antropologicznej" (A trap of anthropological typology).

8. Tomasz Kozłowski, „Refleksje antropologa nad identyfikacją szczątków kostnych znalezionych w Katedrze we Fromborku jako należących do Mikołaja Kopernika. Próba krytycznego spojrzenia" (Reflections of an anthropologist on the identification of bone remains found in the Frombork Cathedral as those of Nicholas Copernicus. An attempt at a critical interpretation).

9. Bronisław Młodziejewski, „Wydolność metod rekonstrukcji wyglądu twarzy w świetle współczesnej antropologii sądowej" (The efficiency of the reconstruction methods of face appearance in the light of contemporary forensic anthropology).

10. Józef Flik, „Szesnastowieczne portrety Mikołaja Kopernika" (The 16th century portraits of Nicholas Copernicus).

11. Tomasz Grzybowski, „Statystyczna i filogenetyczna interpretacja wyników badań mitochondrialnego DNA domniemanych szczątków Mikołaja Kopernika z archikatedry fromborskiej" (A statistical and philogenetic interpretation of research results of mtDNA of the alleged remains of Nicholas Copernicus from Frombork cathedral).

12. Peter Gwozdz, "Copernicus Y-DNA is Haplogroup R1b1b2a1" (in English).

13. Adam Walanus, „Datowanie radiowęglowe, a sprawa grobu Kopernika" (Radiocarbon dating and the case of the Copernicus grave).

14. Lidia Smentek, "Lost OR found?" (in English).

15. Michał Kokowski, „Procedura identyfikacji szczątków nr 13/05 jako szczątków Kopernika w świetle racjonalności uzasadniania i retoryki perswazji" (A procedure of identification of remains no. 13/05 as Copernicus', in the light of rationality of justification and rhetorics of persuasion).

III. *14th Kraków Methodological Conference. Road to Reality with Roger Penrose* (Warszawa 17-18 May, Kraków 20-21 May 2010)

The conference was co-organized by the Mathematical Institute of the Polish Academy of Science, Polish Academy of Arts and Sciences, Jagiellonian University, Center for Advanced Studies of the Warsaw University of Technology, and the Copernicus Center.

The first part of the conference, devoted to Penrose's mathematics and physics, was held in the Institute of Mathematics Polish Academy of Sciences in Warsaw on 17-18 May 2010 and the second part, held in the Jagiellonian University in Krakow on 20-21 May 2010, was devoted to Penrose's physics and philosophy.

The goal of the conference was not only to study Penrose's original contributions to mathematics, physics and philosophy, but also to critically assess his ideas and continue his line of research.

The honorary guest of the conference was Sir Roger Penrose.

During the conference, the following papers were presented (in English):

In Warszawa:

1. Sir Roger Penrose, "Dynamical Equations for the Big Bang"
2. Andrzej Trautman, "The rudiments of twistor theory"

3. Abhay Ashtekar, "The Many Faces of Black Holes"
4. Nick Woodhouse, "Twistors and Special Functions"
5. Iwo Białynicki-Birula, "Can homodyne principle help to understand quantum measurement?"
6. Artur Ekert, "Less reality, more security?"
7. Jerzy Kijowski, "Quasi-local energy of gravitational field"
8. Giuseppe Marmo, "Geometry and Quantum Mechanics"
9. Dominique Lambert, "The mathematical road towards Biology: speedway or Holzweg?"
10. Marek Kuś, "Quanglement and identical particles"
11. Michał Heller, "General Relativity and von Neumann Algebras"
12. Shahn Majid, "A Lie theory of finite simple groups"

In Kraków:

13. Simon Saunders, "Natural necessity: mathematics and structural realism"
14. James Ladyman, "Three claims about symmetries in physics from «The Road to Reality»"
15. Shahn Majid, "Quantum Born reciprocity and the cosmological constant"
16. Józef Życiński, "Plato, Penrose and Ellis: Ontological Platonism in Foundations of Mathematics"
17. Roman Duda, "Deep roots of mathematics"
18. Bogdan Dembiński "Peri ideon logikos"

19. Krzysztof Wójtowicz, "Can empirical facts become mathematical truths"
20. Stanisław Krajewski, "Penrose's metalogical argument is unsound"
21. Donald Lynden-Bell, "Mach's Principle within General Relativity"
22. Gordon McCabe, "The non-unique universe"
23. Leszek Sokołowski, "On the abuse of gravity theories in cosmology"
24. Wojciech Grygiel, "Hawking and Penrose — one physics, two philosophies?"

On May 20, 2010, in Auditorium Maximum of the Jagiellonian University in Kraków, Sir Roger Penrose delivered the first Copernicus Center Lecture entitled "Aeons before the Big Bang".

IV. *Causes and Tenses: Formal Perspectives* (Kraków, September 10-11 2010)

The international seminar "Causes and Tenses: Formal Perspectives" was held on September 10-11, 2010. It was co-organized by the Copernicus Center and the Faculty of Philosophy of the Jagillonian University. The honorary chairman of the seminar was Nuel Belnap of Pittsburgh University.

The underlying idea was to bring together researchers conducting probabilistic causation (analyzed formally) with researchers of tenses, analyzed from either a logical or metaphysical perspective. The emphasis was on small data structures for analyzing causes and indeterminism, common causation and causal completability, and new techniques for the problem of future contingents.

During the conference, the following papers were presented (in English):

1. Miklós Rédei and Zalán Gyenis, "Characterizing common cause closed probability spaces"
2. Michał Marczyk and Leszek Wroński, "Remarks on the notion of causal closedness of classical probability spaces"
3. Nuel Belnap, "How case-intensional semantics prevents the slingshot from hitting its target"
4. Thomas Müller, "Small data structures for representing indeterminism"
5. Fabrice Correia, " A „classical" tempo-modal logic for future contingents"
6. Jacek Wawer, "How to Survive on the Thin Red Line?"
7. Tomasz Placek, "On the assertion problem"

V. *Game Theory and the Law* (Kraków, 9 October 2010)

The international seminar "Game Theory and the Law" was co-organized by the Copernicus Center and the Department for the Philosophy of Law and Legal Ethics of the Jagiellonian University in Kraków.

The papers were presented by: Bartosz Brożek, Bart Du Laing, Arthur Dyevre, Mariusz Golecki, Eunate Mayor, Tomasz Pietrzykowski, Aleksandra Samonek, Giovanni Sartor, Wojciech Załuski.

VI. *The Existence of God: Theism, Agnosticism, Atheism* (Kraków, 3 December 2010)

The conference was co-organized by the Queen of Poland Saint Hedwig's Foundation, the Copernicus Center, and the Pontifical University of John Paul II.

During the conference, the following papers were presented (in polish):

1. Jan Woleński, „Dowody na istnienie Boga i logika" (Proofs of God's Existence and Logic)
2. Marek Porwolik, „Ojca Bocheńskiego formalizacje czwartej drogi św. Tomasza z Akwinu" (Father Bocheński's Formalizations of Aquinas' Fourth *Via*)
3. Jan Hartman, „Kosztowna fikcja" (A Costly Fiction)
4. Mieszko Tałasiewicz, „Agnostycyzm a wiara. Kilka uwag metodologiczno-apologetycznych" (Agnosticism and Faith. Some Methodological-Apologetic Remarks)
5. Damian Wąsek, „Aspekty językowe sporu o istnienie Boga" (Linguistic Aspects of the Controversy Over God's Existence)

At the end of the conference, a panel discussion "Who's Got Proof And Who's Got Reason" was held.

Calendar

6 January 2010 - A lecture in methodology (*Science and the Big Questions*) was delivered by Fr. Dr. Zbigniew Liana.

13 January 2010 - The first lecture in neuroscience (*Science and the Big Questions*) was delivered by Professor Jerzy Vetulani.

14 January 2010 - Michael Heller delivers a lecture ,,Teologia naturalna a współczesna kosmologia" (Natural theology and contemporary cosmology) at the meeting of *Fides et Ratio* Comission.

16 January 2010 - Copernicus Center Colloquium #3.

20 January 2010 - The second lecture in neuroscience (*Science and the Big Questions*) was delivered by Marcin Szafrański.

27 January 2010 - The third lecture in neuroscience (*Science and the Big Questions*) was delivered by Professor Dominika Dudek.

22-23 February 2010 - Conference *The Nicholas Copernicus grave mystery. A dialogue of experts*.

25-26 February 2010 - Conference *The State in Micro- and Macroscale* under the patronage of Copernicus Center, at which Michael Heller delivers a lecture ,,Granice metody naukowej" (The limits of scientific method).

3 March 2010 - A lecture in quantum mechanics (*Science and the Big Questions*) was delivered by Fr. Dr. Tadeusz Pabian.

10 March 2010 - Michael Heller delivers a lecture ,,Czy wszechświat wyjaśnia się wszechświatem?" (Can we explain the universe with the universe itself?) (*Science and the Big Questions*).

17 March 2010 - A lecture in metaphysics (*Science and the Big Questions*) was delivered by Fr. Professor Stanisław Wszołek.

24 March 2010 - A lecture in emergence (*Science and the Big Questions*) was delivered by Dr. Andrzej Koleżyński.

28 April 2010 - Michael Heller delivers a lecture ,,Nauka i Wiara" (Science and Faith) (*Science and the Big Questions*).

5 May 2010 - The first lecture in evolutionary theory (*Science and the Big Questions*) was delivered by Professor Paweł Koteja.

12 May 2010 - The second lecture in evolutionary theory (*Science and the Big Questions*) was delivered by Dr. Paweł Polak.

17-18, 20-21 May 2010 - 14th Kraków Methodological Conference.

18 May 2010 - death of Professor Andrzej Pelczar, a great mathematician, rector of the Jagiellonian University from 1990 to 1993, the President of the Council of the Copernicus Center.

2 June 2010 - A lecture in Bell's inequalities (*Science and the Big Questions*) is delivered by Professor Tomasz Placek.

10-11 September 2010 - Conference *Causes and Tenses: Formal Perspectives*.

6 October 2010 - A lecture entitled ,,Pierwsze konflikty między nauką a religią" (First Conflicts Between Science and Religion) was delivered by Michael Heller (*Science and Religion*).

9 October 2010 - Seminar *Game Theory and the Law*.

13 October 2010 - A lecture entitled ,,Rozum a wiara w pierwszych wiekach chrześcijaństwa" (Reason and Faith in Early Christianity) was delivered by Sr. Dr. Teresa Obolevitch (*Science and Religion*).

20 October 2010 - A lecture entitled ,,Wczesnośredniowieczne próby zmagania się z problemem rozumu w obrębie wiary" (Early Medieval Attempts at the Reconciliation of Faith and Reason) was delivered by Fr. Dr. Zbigniew Liana (*Science and Religion*).

27 October 2010 - A lecture entitled ,,Wielka Scholastyka. Rozum w opozycji do wiary" (The Great Scholasticism: Faith in Opposition to Reason) was delivered by Professor Bartosz Brożek (*Science and Religion*).

3 November 2010 - A lecture entitled ,,Rozum a wiara w renesansie" (Faith and Reason in Renaissance) was delivered by Fr. Dr. Włodzimierz Skoczny (*Science and Religion*).

10 November 2010 - A lecture entitled ,,Czternastowieczne kontrowersje wokół relacji nauki i wiary" (Faith and Reason in the 14th Century) was delivered by Fr. Professor Janusz Mączka (*Science and Religion*).

17 November 2010 - A lecture entitled „Okultyzm a nauka przed Oświeceniem" (Occultism and Science Before Enlightement" was delivered by Fr. Dr. Zbigniew Liana (*Science and Religion*).

24 November 2010 - A lecture entitled ,,Kopernika problemy z Inkwizycją" (Copernicus and the Inquisition) was delivered by Fr. Professor Janusz Mączka (*Science and Religion*).

1 December 2010 - A lecture entitled „Sprawa Galileusza" (The Case of Galileo) was delivered by Fr. Dr. Włodzimierz Skoczny (*Science and Religion*).

3 December 2010 - Conference *The Existence of God: Theism, Agnosticism, Atheism*.

8 December 2010 - A lecture entitled „Powstanie nauki a teologia" (The Emergence of Science and Theology) was delivered by Fr. Dr. Wojciech Grygiel (*Science and Religion*).

15 December 2010 - a panel discussion ,,Rozum a wiara — konflikt czy współistnienie?" (Faith and Reason: Conflict or Coexistence?) was held in Rzeszów (*Science and Religion*).

Essays »

Plato, Penrose, and Ellis: Ontological Platonism in the Foundations of Mathematics

Józef M. Życiński

1. Introduction

Contemporary debates concerning the foundations of mathematics imply new versions of mathematical Platonism and suggest that forthcoming changes in meta-mathematical philosophy could provide breakthrough discoveries in modern physics bringing insights "more wonderful (...) than those we have been blessed with in the 20th century"[1]. I would like to illustrate this renewal of Plato's influence in the philosophy of mathematics by bringing to our attention the new non-equivalent ideas developed by Roger Penrose and George F.R. Ellis. The differences between these two authors may turn out to be heuristically inspiring when the search for the physical Theory of Everything suggests important epistemological transformations in contemporary science. In an attempt to overcome them, in his polemic with Nancy Cartwright, Penrose does not exclude that "the physicist's ultimate goal of a completely unified picture is an indeed unattainable dream"[2].

The essence of Roger Penrose's ontological Platonism is expressed in the acknowledgement of the objective existence of mathematical objects. As he argues, the Mandelbrot set was not invented by mathematicians but only discovered in a similar way to the discovery of the planet Neptune. In such claims, "the objective existence" seems to be the most important expres-

1 Roger Penrose, *The Road to Reality. A Complete Guide to the Laws of the Universe*, A. Knopf, New York 2005, p. 1045. Further cited as RR.

2 "Response by R. Penrose", [in:] *The Large, the Small and the Human Mind*, Cambridge University Press 2000, p. 180.

sion. Penrose, together with another famous representative of relativistic cosmology, George F. Ellis, argues that "to be" means "to have a causal effect in the world of physical particles and forces"[3]. Karl Giberson, considering the strong ontological commitment of the author of *The Road to Reality*, expresses his conviction that such a commitment makes Penrose "Plato's greatest living champion". The authors who reject metamathematical Platonism and try to emphasize the role of pragmatic factors in the foundations of mathematics counter argue that "Penrose's biggest mistake is to be a hard core Platonist".

The differences in the appraisal of the philosophical ideas expounded by the author of *The Road to Reality* are so deep that, on the one hand, he is criticized as a "religious fanatic"[4] who denies the novel ideas of the "New Atheism", on the other his *Emperor's New Mind* is regarded as a classic exposition of New Age ideology.[5] The radical difference in these similar appraisals demonstrate how important and complicated the philosophical issues are in the ambitious attempt undertaken by our author to understand the foundations of mathematics and to determine the ontological structure of the world of nature.

2. Three Worlds Ontology

In a style analogous to the classical Karl Raimund Popper's arguments, Penrose distinguishes three levels of ontological existence. Their nature is to be appropriately: Platonic-mathematical, physical, and mental-conscious. They cannot be reduced to each other, since the importance of algorithmic

3 G.F.R. Ellis, "On the Nature of Emergent Reality", [in:] *The Re-Emergence of Emergence*, P. Clayton, P. Davies (ed.), Oxford University Press, Oxford 2006, p. 85.

4 Richard Henderson, *Mistakes*

5 Ben Best, F.A. Wolf

reasoning, so important for reductionist arguments, must be questioned if one recognizes the epistemological significance of Gödel's incompleteness theorem. In his critique of metamathematical formalism, which seemed justified in the philosophy of mathematics in the epoch when a young Bertrand Russell prepared *Principia Mathematica* together with Alfred N. Whitehead, Penrose declares: "I believe that the 'Gödelian' case is, at root, really a very powerful one, even though some people seem most reluctant to take it on board"[6]. Consequently, he does not believe that the entire domain of human logic could be interpreted algorithmically. Following John Searle's critique, in which one tries recursively to interpret human understanding, Penrose argues that the standpoint of strong AI seems to drive one into an extreme form of dualism, since it suggests that the very reality of algorithms form the 'substance' of human thoughts, feelings, and conscious perceptions.[7]

Consistently, the author of *The Road to Reality* disagrees also with the views of Hawking expressed in his famous paper *Is the end of physics in sight?*. He is convinced that the important discoveries both in physics and in biology will be provided by science in our century. The importance of these discoveries will be great, because the standard laws of physics in their present form cannot be relevant for the most important problems of contemporary biology.[8] The physics of the 21st century can bring groundbreaking discoveries that will allow us to find answers to many questions now believed to pertain to the realm of mystery. An essential change in the cognitive perspective may, in his view, lead to the overcoming of the present impasse in super-

6 "Response by R. Penrose", p. 175.

7 R. Penrose. *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics*, Oxford University Press, New York 1989, p. 21.

8 "Response by R. Penrose", p. 182.

string physics and to the development of radically different approaches. In particular, he believes that the future fundamental theory that "lays claim to any kind of completeness must also have the intellectual potential to explain the nature of conscious mentality"[9]. Computational functionalism claims that it is merely computational activity (of an unspecified activity) that gives rise to conscious mentality. He does not agree with this view and says that despite his personal inclination towards Platonism in the foundations of mathematics, he does not accept the claim that 21st century physics would be able to provide solutions to the great philosophical problems. The latter may be classified, with John Barrow, as the so-called prospective problems. They are related to properties that can be neither enumerated nor calculated; they cannot be identified nor generated by a sequence of logical steps.[10]

Beauty, simplicity, truth are examples of prospective properties. They cannot be exhausted in the mathematicized descriptions which the natural sciences can provide. No programme or formula can generate all of what constitutes beauty or ugliness. However, in Penrose's works on the role of mathematics in science or on the objective status of the relations it studies, there often appear references to categories such as beauty, mystery or the miraculous connection between mathematics and physics.[11] The prospective properties of things cannot be forced into the framework of any logical Theory of Everything. A non-poetic account of the world cannot be complete.[12] The further development of our knowledge will not automatically overcome cogni-

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9 RR, p.1033.

10 John Barrow, *op. cit.*, p. 271.

11 As an example of "most profound mystery" Penrose mentions spacetime singularities in black holes. RR, p. 1045.

12 *Ibidem*, p. 272.

tive difficulties whose importance was indicated for example by Kurt Gödel.

When trying to answer the question of "What is reality?", in his philosophical interpretation of the hierarchy of cosmic structures, Penrose describes the physical World 2 as emerging out of the timeless world of mathematics, the World 1. The deeper we understand the laws of physics, the more we are driven into this world of mathematics and mathematical concepts. Its mental image we find in World 3. Each 'world' could be described as its own specific kind of existence. This specificity cannot, however, be interpreted in terms of ontological isolation, because there are several important questions which require reference to more than one world in the process of diachronic growth of science. Here important questions emerge: How does mentality come about in the physical structures with which it is associated? How should quantum theory be related to the mind-body problem? It seems impossible to answer such questions without reference to the emergence, supervenience, downward causality. The very concept of emergence remains, however, semantically fuzzy; it seems to imply the irreducibility of the world 3 to the lower levels of the world 2 and 1. The essence of this irreducibility brings many problems discussed, among other authors, by G.F.R. Ellis when he refers to the Biblical terms of God's *kenosis* and tries to explain some phenomena regarded by Penrose as mysterious ones. Both authors refer to Plato's ontology, their understanding of Platonism profoundly differs and can be subjected to rational appraisal.

Penrose attributes the Platonic structure not only to objects of mathematics that express the absolute Truth, but also to "the other 'Platonic absolutes' of Beauty and Morality". He claims, however, that "the whole issue of Morality is ultimately dependent upon the 'World of Mentality'". Each of the three worlds:

'Platonic-mathematical, physical, and mental — has its own kind of reality,' but the Platonic world may be the most primitive of the three, primitive in the sense of ontological primordiality. In his exposition of the essence of Plato's theory of ideas, W.D. Ross claims: "the essence of the theory of Ideas lay in (...) recognition of the fact that there is a class of entities, for which the best name is probably 'universals', that are entirely different from sensible things." (Ross, 1952, p. 225). To eliminate the psychological undertones in which ideas are always understood as a product of the reflection of a particular mind, many authors prefer terminology in which the word "idea" is replaced by the word "form". R.C. Cross and A.D. Woozley justify such terminological preferences when they refer to Plato's ontology as to the Theory of Forms. They argue that "the English word 'form' keeps near to the meaning of the Greek words, and is also free from misleading associations of the word 'idea" (Cross, Woozley, 1978, p. 82).

Different versions of the theory of Forms are proposed in different writings by Plato. Their interpretations by particular historians of philosophy also differ deeply. I would refer to the expression of *Parmenides* 132 D, where the Forms are presented "as it were patterns fixed in the nature of things". A similar understanding of Forms is presented in *Philebus*. According to this exposition, Forms are paradigms and patterns by which all sensible particulars are given names and identified as much as particulars participate in these Forms (Sayre, 1983). In his comment to the paradigmatic role of the Forms, Peter Geach refers to the standard yard as to illustration of self prediction. One should here distinguish between a universal pattern of the standard yard and its physical particular actualization. The cognitive problem of our generation is that fascinated by particulars we do not appreciate enough the role of universals in the physical structure of the universe. Only the universal language of mathematics helps us be open to this structure of the world that manifests itself through the laws of nature. The differences in opinions in the interpretation of Plato point out that both Penrose and Ellis could seek their own understanding of the Platonic forms in the context of contemporary science. The essence of this attitude can be expressed by the acknowledgement that in our physical world of changing parameters and fluent appearances there is a stable factor, a pattern fixed in the nature of particular objects and of evolving systems. Plato describes these factors calling them "real", "completely real", or "truly existent". In the language of contemporary philosophy these truly existent constituents of the world are called universals. Some universals could refer to the non-physical aspects of reality, e.g., to the existential, aesthetic or ethical dimension of human life. The problem remains how to interpret in detail the role of Platonic forms in contemporary scientific theories.

3. Ellis' ontological pluralism

In an alternative version of Platonism, G.F.R. Ellis, akin to Penrose in his cosmological research, distinguishes five levels of reality, irreducible to one another. They are constituted, respectively, by:

World 1: The world of physical objects, of energy, particles, etc.

World 2: The world of individual and social consciousness — it comprises mental processes, ideas, emotions, social legislation and conventions.

World 3: The world of possibilities understood in Aristotelian sense. Some of these possibilities actually exist; they constitute the physical world 1.

World 4: The Platonic world of abstract relations and structures — e.g., rational and irrational numbers, spinors, symmetry groups, supervector spaces, the Mandelbrot set, variation principles, Hilbert spaces, Dirac's equation, etc.[13] In their essence they are independent both from mental processes of human beings (World 2) and from their particular actualizations in course books or in physical objects (World 1). Pythagoras Theorem would still to be valid even if a nuclear catastrophe destroyed all the course books explaining it and killed all the creatures that understand what the right-angled triangle in Euclidean geometry is.

Apart from mathematical forms, Ellis includes in World 4 both universal formulas of the laws of nature and the contents of concepts playing a fundamental role in contemporary science. He is also inclined to place in World 4 aesthetic forms that are the basis for our perception of beauty at the level of World 2. Referring to the research practice of contemporary physics, Ellis observes that while for primitive man World 1 objects were particularly important, for scientists a fundamental role is played by the objects of World 4. The simplest assumption would be that they are creations of human mind produced by generalization of the previously applied study procedures, later proved extremely useful in describing new types of phenomena discovered in nature. Ellis does not agree with such an interpretation and, together with Penrose[14], emphasizes that the world of mathematical relations constitutes the fundamental structure of the world.

Metaworld 0: The Word of God, underlying reality of worlds 1-4. It interacts with all other worlds but remains transcend-

13 *Natures*, p. 337.

14 Ellis quotes especially two works by Penrose: *Shadows of the Mind*, Oxford University Press, Oxford 1989, and *The Large, the Small, and the Human Mind*, Cambridge University Press, Cambridge 1997. His views are also akin to the philosophy presented by Eugene Wigner in a classic paper "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", *Communications in Pure and Applied Mathematics*, 13 (1960) 1.

ent to them. The deepest ontic justification of all the processes and objects from worlds 1-4 is situated at this level. This is the world that allows us to understand the reality of reason, value, and meaning. In theology the ontic basis of this reality is called God. In various philosophical systems, however, different terms are used: from the transcendent cause to unmoved mover to the highest good and beauty.

In looking for an ontological justification for World 4 as well as for Worlds 1-3, Ellis declares the necessity of accepting the reality of Metaworld 0. Without referring to its reality, it is impossible to answer many questions formulated at the level of other worlds. In particular, the reference to World 0 constitutes the axiological and ethical dimension of events, while the reference to individual human thought, i.e. World 2, remains the most important for epistemological aspects. In its phylogenetic development, the *Homo sapiens* species first became fascinated with World 1. The concern about stones, bows, cars and banks played a very important role in the evolutionary struggle for survival. World 2, constituted by individual thought, led to considerations that would not necessarily have to facilitate the struggle for survival. Admiration for Mozart's music or excessively intense reflection on infinite sets, as in the case of Georg Cantor, could have even put survival at risk. Our generation travelled the blind alleys of this world by escaping into virtual reality and into the world of intense sensations which lead to addictions. Our future development depends on the use of potentialities hidden in World 3 and from the further discovery of the Platonic structure of World 4.

The ontic reason for this pluralist ontology is that God corresponds to the reality of World 0. His ontological *kenosis* liberates us from an ambitious search for perfect structures by introducing the mark of incompleteness into ontology, like Gödel's incompleteness theorem introduces it into rich logical systems. An ascent to the heights of thought, combined with a search for the ultimate reason of rationality found in nature, leads to the discovery of God's subtle presence in everything that exists. At the level of His presence we reach the ultimate conditions of meaning and of the sense of existential fulfilment in the rational, pluralist universe.

In Ellis' arguments, the emergence of higher levels of reality from less complicated lower structures is based on non-linear relations between components. The new emerging properties, which do not occur at the earlier process of evolutionary development, emerge then from the lower level structures and should be explained by reference to the concept of supervenience and downward determination. In this framework, the non-physical reality of information and end-directed processes has physical effects in the world of forces and particles; consequently it must be recognized as real[15].

Along with the traditional questions of the cause or reason for the existence of accidental physical objects of World 1, the development of knowledge brings many new questions, in which new variants of the classical dilemma of Leibniz recur: Why is there something rather than nothing? The questions that are more significant to Ellis — to a great extent thanks to his interest in theoretical physics — and which Leibniz's formulation includes are: Why there are universal laws in nature, while it might well be an uncoordinated chaos, impossible to grasp either by mathematical formulas or any general concepts? How can we explain the rationality of nature expressed in the validity of variational principles? How are we to understand enigmatic

15 G.F.R. Ellis, "True Complexity and its Associated Ontology", [in:] J. Barrow, C.L. Harper (eds.), *Science and Ultimate Reality From Quantum to Cosmos*, Cambridge University Press 2003.

coincidence — described by the anthropic principle — of physical parameters, without which the emergence of protein life forms would be impossible[16]? What can explain the "congruence" of mathematics from World 4 with the objects of World 1; they are, in fact, two entirely different worlds, and there are no reasons why twistors and spinors should allow for the conditions occurring at the level of stones or chairs? Those questions concerning the ultimate reason that constitutes both the existence of the world and our knowledge of it, leading to the recognition of God as the factor that renders the processes and objects of Worlds 1-4 free of contradiction.

4. Mystery or *kenosis*?

In his argument Ellis, to a much greater degree than Penrose, refers to the role of religious experience. Where Penrose falls silent and speaks of an unspecified role of mystery, Ellis turns to biblical concepts in which the essence of God's influence on the world is expressed in the so called *kenosis*, the self-abasement of God — an action remote from victorious triumphalism and bearing a mark of subtle goodness, solidarity and humility of God cooperating with man.

Kenosis, underlying the basis of deep ethics, according to Ellis, denotes unlimited love, expressed in the acceptance of suffering and self-sacrifice to reveal human spiritual generosity. In human life, its classical example could be found in the life of Mahatma Gandhi or Mother Theresa of Calcutta. In the kenotic perspective, the rise of human spirituality could be explained through a slow evolutionary process underlying the historical

16 The issue is discussed by Ellis in his paper "The Theology of the Anthropic Principle", [in:] *Quantum Cosmology and the Laws of Nature*, R.J. Russell et al. (ed.), Vatican City State: Vatican Observatory 1993, pp. 367-405.

emergence of complexity; ethics constitutes the highest level in the hierarchy of structures that emerge in cosmic evolution. Some critics of such interpretations try to reject cognitive realism in science, ethics and philosophy of religion. They defend a radical opinion that Plato's ontology in Penrose's arguments is equally valid as creationist science. It is impossible to support such an appraisal in the context of contemporary scientific investigations. Creationist science implies pseudo-explanations inconsistent with the research practice of contemporary biology and physics. Penrose's ontological proposals are generated by an attempt to discover implicit presumptions underlying the research practice of contemporary mathematics. The authors who do not like ontological Platonism proposed by Penrose and Ellis, try to propose cognitive irrealism as an explanatory alternative. The consistent denial of cognitive realism brings however rejection of explanatory values of all philosophical theories, among them the theories that call scientific realism into question.

When explaining the process of evolutionary growth, Ellis distinguishes upward and downward interactions in the hierarchy of structures. The flow of information between different levels of structures is essential to the distinction between the top-down (i.e. downward) and bottom-up (upward) interactions in the evolutionary process. The study of this relation is enabled by the fact that properties absent at the lower levels of structural complexity emerge at the higher levels; their emergence is not liable to reductionist explanation referring to their components, as this type of explanation includes only bottom-up interactions. G.R.F. Ellis believes that the traditional reductionism could be regarded as a symptom of fundamentalism, the former being an attempt to reduce human personality, morality, culture to the level of physical-biological phenomena[17]. The holistic approach to investigating the top-down influence on the whole evolving system is still criticized by those physicists whose cognitive attention is absorbed by the functioning of strictly deterministic laws.

5. Overcoming ontological pluralism?

To settle the argument of whether we must distinguish in the ontological structure of the universe five mutually irreducible levels (Ellis), or only three levels (Penrose, Popper), it would be necessary to precisely define the concepts of emergence, continuity, supervenience, reducibility. In contemporary philosophy those concepts assume deeply varying senses. Therefore, setting aside the differences between them, common elements in different versions of Platonism in understanding of the foundations of mathematics should be noted. They indicate that:

1. The effectiveness of contemporary mathematics in describing natural processes cannot be reduced to the level of its pragmatic applications. There is the basic level of ontological structures that makes it possible to explain this effectiveness rationally. The conviction, however, that the future development of physics will lead to solving all the difficult problems, is an expression of cognitive optimism. John Ellis from CERN, who provocatively called a future theory of unification in physics the Theory of Everything, regrets today that he did not call it a Theory of Nothing, which would spare us the naive ideological comments made about it. When, at the level of metaphors, the difference between "everything" and "nothing" has been

17 G.F.R. Ellis, "On the Nature of Emergent Reality", [in:] *The Re-Emergence...,* p. 104.

obliterated, one can easily justify any statement. From zero anything can be inferred. This is the direction that George Ellis's argument takes when he stresses that the present lack of answers to fundamental questions results from insufficiently advanced level of research that will be overcome after some future discoveries. He emphasizes: "no future scientific advances will change" the present situation. Some, ethical, aesthetic and metaphysical issues, lie outside of cognitive competence of science and they would be never solved scientifically[18].

2. There is no universally accepted version of mathematical Platonism that would allow a unified, generally recognized version of philosophy of God. At the level of philosophy of mathematics it is difficult to justify the concept of God as a person; however, various interpretations of Plato's views are a normal phenomenon in hermeneutics. As a result, interpretations claiming the reality of Platonic forms are admissible, while their identification with God of the Christian theism is treated with reserve.

3. The future development of research on the Theory of Everything may provide answers to many questions concerning the ontological status of mathematical objects. It should not be expected, however, that the 21st century would give a definitive answer to numerous problems that we were not capable of solving during the previous twenty centuries. The clear perception of certain questions does not imply their definitive settlement.

4. In studies that aim to overcome the present limitations, it is essential to go beyond the current language barriers.

18 EMolB, p. 256.

This is being done by developing the new terminology in noncommutative geometry, defining various concepts of emergence, specifying the relations obtaining between supervenience and subvenience, taking into account the downward causation in evolutionary processes.

5. Earlier equivalents to the proposals by Penrose and Ellis can be found in philosophical theories of many other authors. In Poland Jan Łukasiewicz, among others, a well-known representative of the Polish School of Logic, argued that the reality of ideal mathematical structures independent of human experience could be regarded as an expression of God's presence in nature (Łukasiewicz, 1937). In his approach, the Penrosian objective mathematical objects are described as a field of formal structures, universal relations, rational interdependencies; this reality in various philosophical schools is presented as the matrix of the universe, the field of rationality, the formal field, the Logos, the Absolute, etc.

6. Regardless of our terminological preferences, on the one hand in this formal structure we find a basic level of physical reality where the amazing effectiveness of mathematics can be explained. On the other hand, in this formal constituent of nature we find the factor which inspires our astonishment and directs our attention towards the theological and philosophical aspects of nature. Theologians and scientists must not be in conflict but, following different methodological rules, they could look for an integral vision of nature.

The Mathematics of the Transcendental Ego

Bartosz Brożek
Adam Olszewski

1. Kant for dummies

Textbook accounts of Kant's philosophy of mathematics run roughly as follows. Kant's grand critical project stems from the observation that is seems *prima facie* impossible to reconcile two particular facts. On the one hand, we have the profound intuition, confirmed by numerous great thinkers, that mathematical knowledge is universal and necessary (the same holds for 'pure' Newtonian physics, some moral precepts etc.). On the other hand, as Hume brilliantly pointed out, there is nothing in our sensual, empirical experience that would validate the claims of universality and necessity which are attributed to mathematical, moral or physical statements. On the contrary, if one assumes the Lockean conception of human mind as *tabula rasa*, there is almost no escape from the conclusion that mathematics, physics and morality are collections of contingent, unjustified theses. 'Almost', as we can always claim with Locke and Hume himself, that mathematical theses are analytic, and are thus safeguarded from the Humean-like attack (the same strategy cannot be applied to physics and morality, of course). However, to say that a mathematical statement, such as "2 + 7 = 9", or "a topological space is normal if and only if any two disjoint closed subsets can be separated by a function", are analytic, amounts to declaring them 'tautological', true on grounds of their meaning or, speaking loosely, conveying no information. This is troublesome: is Fermat's last theorem 'tautological' or devoid of meaning if its proof required so much effort and imagination? The dilemma we face is the following: either mathematical truths are universal and necessary but analytical, or they are purely contingent.

Kant's famous solution to this dilemma lies in acknowledging that, apart from analytic *a priori* and synthetic *a posteriori* judgements, there are also synthetic *a priori* judgements. They are possible — and here is the crux of Kant's argumentation — because the knowing subject or the transcendental ego is not a passive recipient of the empirical data; it actively filters them through its transcendental structures. In other words, any act of cognition is a fusion of what comes from the *noumena* (the things in themselves) and from the transcendental ego (the subject). According to Kant, the structure of the transcendental ego includes pure intuitions of time and space, the categories and the regulative ideas. Mathematical theses are justified on the grounds of pure intuition. Thus, they are synthetic *a priori*: they are synthetic as they expand our knowledge, and they are *a priori* as they are not based on sensual experience but on the pure intuitions of space and time. In a way, mathematical statements are descriptions of space and time, when space and time are not aspects of the world which is independent of the subject, but are the elements of the subject itself.

Kant's mathematical subject is the *transcendental* ego. What does it mean? The term 'transcendental' was used by medieval thinkers to refer to some properties that were predicated of every entity. Thus *bonum*, *pulchrum*, *ens*, *unum*, *verum* were *transcendentalia*. Later, Wolff altered the understanding of 'transcendental'. Even though he still used it in its ontological meaning, he also employed it in epistemology: 'transcendental' is any property that can be ascribed to each and every act of cognition. It is in this meaning that Kant uses 'transcendental'. Transcendental conditions of cognition, such as the intuitions of space and time, or the categories, are the conditions of each and every act of cognition. Cognition is impossible outside of the structure of the transcendental ego. The transcendental ego itself is not something we can comprehend. As Kant puts it:

> space itself, however, together with time, and, with both, all appearances, are not things, but rather nothing but representations, and they cannot exist at all outside our mind; and even the inner and sensible intuition of our mind (as an object of consciousness), the determination of which through the succession of different states is represented in time, is not the real self as it exists in itself, or the transcendental subject, but only an appearance of this to us unknown being, which was given to sensibility. The existence of this inner appearance, as a thing thus existing in itself, cannot be admitted, because its condition is time, which cannot be a determination of any thing in itself.[1]

Therefore Kant claims that the transcendental ego cannot be given as an object of experience. It follows that the transcendental subject cannot — for logical reasons — be studied with the tools offered by psychology. It is not the 'psychological self', but rather what — so to speak — lies 'beneath' it. The question is, therefore, how can one distil the features of the transcendental ego, how to investigate it, when — *ex definitione* — any empirical method fails in this respect?

Kant makes a tripartite distinction between the types of cognition or the types of the principles of cognition:

> we will call the principles whose application stays wholly and completely within the limits of possible experience immanent, but those that would fly beyond these boundaries transcendent prin-

1 I. Kant, *Critique of Pure Reason*, translated by P. Guyer and A.W. Wood, Cambridge University Press, Cambridge 1998, A492. This is the edition we shall quote below by identifying the section of the *Critique* to which the quoted passage belongs.

> ciples. But by the latter I do not understand the transcendental use or misuse of categories, which is a mere mistake of the faculty of judgement when it is not properly checked by criticism, and thus does not attend enough to the boundaries of the territory in which alone the pure understanding is allowed its play; rather, I mean principles that actually incite us to tear down all those boundary posts and to lay claim to a wholly new territory that recognizes no demarcations anywhere. Hence transcendental and transcendent are not the same. The principles of pure understanding we presented above should be only of empirical and not of transcendental use, i.e., of a use that reaches out beyond the boundaries of experience. But a principle that takes away these limits, which indeed bids us to overstep them, is called transcendent. If our critique can succeed in discovering the illusion in these supposed principles, then those principles that are of merely empirical use can be called, in opposition to them, immanent principles of pure understanding.[2]

Thus, Kant distinguishes between three kinds of cognition. One can speak of immanent cognition, i.e. one that stays within the limits of possible experience. Still, there is transcendent cognition, which reaches beyond those limits; as such, it is ungrounded: it brings no knowledge, but illusion of knowledge. Further, there should be distinguished transcendental cognition, one that pertains to the determination of the limits of possible experience. This mode of thinking is fully admissible. Indeed, the entire enterprise of the *Critique of Pure Reason* consists of transcendental argumentation; it is, as Kant puts it, "occupied not so much with objects but rather with our *a priori* concepts of objects in general."[3]

The method Kant utilizes in order to uncover our *a priori* concepts is that of transcendental deduction. It follows, boldly

2 B252-253.

3 B25.

speaking, a general scheme of reasoning which can be reconstructed as:

(1) People have a certain experience (e.g., mathematical).
(2) *C* is the necessary condition for the experience to occur (e.g., the utilization of the pure intuitions of time and space is necessary for the mathematical experience to emerge).
Therefore: (3) *C* (e.g., there exist pure intuitions of time and space).

From a formal point of view the argument can be reconstructed in modal logic. Let *E* stand for the experiences in question and *C* for the necessary condition thereof. The argument has the following form:

(1) E	(premise)
(2) $\Diamond E$	(from (1), by $\Diamond$ introduction)
(3) $\Diamond E \rightarrow \Box C$	(premise)
(4) $\Box C$	(from (2) and (3) by *modus ponens*)
(5) C	(from (4) by $\Box$ elimination)

It is common, however, to reconstruct transcendental arguments in a slightly different form:

(1*) E	(premise)
(2*) $\Diamond E$	(from (1*), by $A \rightarrow \Diamond A$)
(3*) $\neg\Box C \rightarrow \neg\Diamond E$	(premise)
(4*) $\Box C$	(from (2*) and (3*) by *modus tollens*)
(5*) C	(from (4*) by $\Box$ elimination)

Logically the difference between the two formulations is unimportant, as (3) and (3*) are equivalent:

(6) $(\Diamond E \rightarrow \Box C) \Leftrightarrow (\neg\Box C \rightarrow \neg\Diamond E)$[4]

4 It is noteworthy that this equivalence is valid in classical logic, but not in intuitionistic logic. However, even in light of Kantian philosophy, it is not troublesome, as for Kant transcendental reasoning is not 'constructive' (unlike the mathematical reasoning).

This is a rough sketch of Kant's method. Such a general perspective is instructive, but fails to account for some essential and intriguing aspects of Kant's philosophy of mathematics.

2. The Euclidean paradigm

It is our thesis that Kant's conception of the mathematical subject is — in many respects — 'tailored' to fit Euclidean geometry. In order to substantiate this thesis we shall focus on two simple examples from Euclid's *Elements*. First, however, let us have a look at the conceptual foundations of Euclidean geometry.

As it is well known, Euclid distinguishes between axioms, postulates and definitions. It is crucial to realize that the Euclidean understanding of these terms differs substantially from their contemporary meaning. Moreover, even among the Ancient writers, there was no agreement as to these distinctions. A good illustration of this fact may be found in Proclus' commentary on Euclid's *Elements*. Proclus identifies three different ways of distinguishing between axioms and postulates. First, he says that

> they differ from one another in the same way as theorems are distinguished from problems. For, as in theorems we propose to see and determine what follows on the premises, while in problems we are told to find and do something, in like manner in axioms such things are assumed as are manifest in themselves and easily apprehended by our untaught notions, while in the postulates we assume such things as are easy to find and effect (our understanding suffering no strain in their assumption), and we require no complication of machinery.[5]

5 *Procli Diadochi in primum Elementorum Euclidis librum Commentarii*, G. Friedlein (ed.), Teubner, Leipzig 1873, 178:12-179:8; quoted after the trans-

Thus, on this reading, axioms are self-evident, while postulates are very simple, easily provable theorems. The second stance is as follows:

> Others again will say that postulates are peculiar to geometrical subject-matter, while axioms are common to all investigation which is concerned with quantity and magnitude. Thus it is the geometer who knows that all right angles are equal and how to produce in a straight line any limited straight line, whereas it is a common notion that things which are equal to the same thing are also equal to one another, and it is employed by the arithmetician and any scientific person who adapts the general statement to his own subject.[6]

The idea is, then, that postulates are domain-specific (they pertain to geometry only), while axioms are generally valid. Finally, there is a view that Proclus attributes to Aristotle: "everything which is confirmed by a sort of demonstration will be a postulate and what is incapable of proof will be an axiom."[7] It is commonly accepted that Proclus misrepresents Aristotle's thought here. As Thomas Heat rightly observes, Aristotle distinguished between axioms (or common notions), postulates, hypotheses and definitions.[8] In *Posterior Analytics*, the Stagirite claims that every demonstrative science must start with indemonstrable, self-evident principles. Some of them (axioms) are common to all sciences. There are also principles characteristic of particular disciplines. Among them one should distinguish between hypotheses and postulates. Hypotheses are propositions which a teacher assumes without proof, although they are a matter

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lation in *The Thirteen Books of Euclid's Elements*, translated by T.L. Heath, Cambridge University Press, Cambridge 1908.

6 *Ibidem*, 182:6-14.

7 *Ibidem*, 181:21 — 182:13.

8 *The Thirteen Books of Euclid's Elements..., op. cit.*, p. 117ff.

of proof. Postulates are likewise propositions, but such that are usually contrary to the opinion of the learner. Finally, definitions are not propositions, as they do not assert the existence or non-existence of anything.[9] Heath believes that it is the Aristotelian conceptual scheme that characterizes best the Euclid's use of the terms involved.

Be that as it may, there is no doubt that Euclid's utilization of the notions of axioms and postulates is different from our understanding thereof. To substantiate this thesis, it is enough to have a closer look at Euclid's axioms and postulates themselves. The axioms include:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

The postulates are:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That is a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles.[10]

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9 Cf. *Posterior Analytics*, I.6, 74b5; I.10.76a 31-77q4.

10 Cf. *The Thirteen Books of Euclid's Elements, op. cit.*

It is interesting to compare the Euclidean understanding of axioms and postulates with Kant's treatment of axioms. Kant says:

> [axioms] are synthetic *a priori* principles, insofar as they are immediately certain. Now one concept cannot be synthetically yet immediately combined with another, since for us to be able to go beyond concepts a third, mediating cognition is necessary. Now, since philosophy is merely rational cognition in accordance with concepts, no principle is to be encountered in it that deserves the name of axiom. Mathematics, on the contrary, is capable of axioms, e.g., that three points always lie in a plane, because by means of the construction of concepts in the intuition of the object it can connect predicates of the latter *a priori* and immediately. A synthetic principle, on the contrary, e.g., the proposition that everything that happens has its cause, can never be immediately certain from mere concepts, because I must always look around for some third thing, namely the condition of time-determination in an experience, and I could never directly cognize such a principle immediately from concepts alone.[11]

This passage encapsulates a conception of axioms that differs from the Aristotelian view. Similarly to Aristotle, Kant believes axioms to be self-evident or certain statements. However, he claims — unlike the Stagirite — that axioms are domain specific: they are characteristic of mathematics alone. In other words, axioms are not propositions which apply to different fields of knowledge. Kant's justification for this thesis rests on his general view of cognition. For a judgement to be self-evident, it must be based on immediate intuition. It follows that only mathematical truths, which are grounded on the pure intuitions of time and space, are deserving of the label 'axiom'. One would be mistaken, however, to claim that Kant's rendering of 'axioms' is similar to

11 B761-762.

our understanding of the term. Contemporaneously, axioms are considered arbitrarily chosen propositions which serve as the starting-points of a deductive chain which leads to the establishment of new theorems. Such a modern deductive conception of mathematical proof is alien to the Kantian philosophy. Kant notes:

> Only apoidictic proof, insofar as it is intuitive, can be called a demonstration. Experience may well teach us what is, but not that it could not be otherwise. Hence empirical grounds of proof cannot yield apoidictic proof. From *a priori* concepts, however, intuitive certainty, i.e. self-evidence, can never arise (…). Thus only mathematics contains demonstrations, since it does not derive cognition from concepts, but from their construction, i.e. from the intuition that can be given *a priori* corresponding to the concepts.[12]

The essential idea behind the quoted passage is that mathematical demonstrations proceed through *construction* rather than a chain of arguments. In order to see what this conception amounts to, let us come back to Euclid's geometry and have a look at two examples.

The first proposition of book I of *Elements* reads: "To construct an equilateral triangle on a given finite straight line." The proof of this proposition is as follows (see Fig. 1). Let AB be the given finite straight line. The goal is to construct an equilateral triangle on the straight line AB. We begin by describing the circle BCD with centre A and radius AB. Next, we describe the circle ACE with centre B and radius BA (in both cases we use Postulate 3, "To describe a circle with any centre and radius"). Consequently, we join the straight lines CA and CB from the point C at which the circles cut one another to the points A and B (here, we use Postulate 2: "To draw a straight line from any point to any point"). Now, since the point A is the centre of the circle CDB, therefore

12 B763.

AC equals AB. Moreover, since the point B is the centre of the circle CAE, therefore BC equals BA (here, the definition I.Def.15 is utilized: "A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another."). Since AC was proved equal to AB, therefore each of the straight lines AC and BC equals AB. And things which equal the same thing also equal one another, therefore AC also equals BC (by the common notion C.N.1: "Things which equal the same thing also equal one another"). Therefore the three straight lines AC, AB, and BC equal one another and thus the triangle ABC is equilateral, and it has been constructed on the given finite straight line AB (by definition I.Def.20: "Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal").

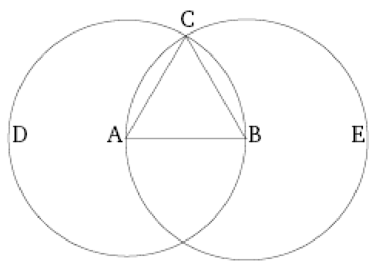

Fig. 1. The construction of Euclid's proposition I.1.

Another example is Proposition 32 of the Book 1 of *Elements*: "In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles." The proof runs as follows (See Fig. 2): Let ABC be a triangle, and let one side of it BC be produced to D. The exterior angle ACD equals the sum of the two interior and opposite angles CAB and ABC, and the sum of the three interior angles of the triangle ABC, BCA, and CAB equals two right angles. Now, we draw CE through the point C parallel to the straight line AB (on the basis of proposition I.31). Since AB is parallel to CE, and AC falls upon them, therefore the alternate angles BAC and ACE equal one another (proposition I.29). Again, since AB is parallel to CE, and the straight line BD falls upon them, therefore the exterior angle ECD equals the interior and opposite angle ABC (proposition I.29). But the angle ACE was also proved equal to the angle BAC. Therefore the whole angle ACD equals the sum of the two interior and opposite angles BAC and ABC. Add the angle ACB to each. Then the sum of the angles ACD and ACB equals the sum of the three angles ABC, BCA, and CAB (common notion C.N. 2). But the sum of the angles ACD and ACB equals two right angles. Therefore the sum of the angles ABC, BCA, and CAB also equals two right angles (proposition I.13 and common notion C.N. 1). Therefore in any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

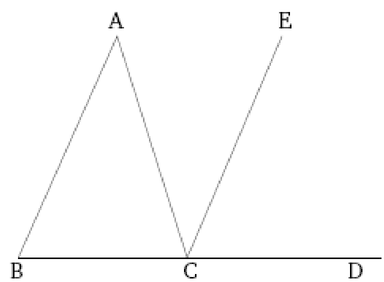

Fig. 2. The construction of Euclid's proposition I.32.

The surprising fact about these two proofs is that — on the basis Euclid's postulates and the modern logic alone — it is impossible to deduce them. In particular, it is often pointed out that the proof of Proposition I.1 is lacking: Euclid's postulates give no guarantee that the point C exists, i.e. that the two circles in fact intersect. The proof of the Theorem I.32, in turn, refers to the propositions I.13, which refers to I.11, which is based on I.1. Thus, the deficiencies of the proof of I.1 undermine also the validity of the proof of I.32.

This problem has generated a substantial literature[13]. In an intriguing essay, *The Method of Analysis*, Jaakko Hintikka and Unto Remes provide an interpretation of some treaties of Pappus.[14] They point out that for the Ancient mathematician proving a geometrical theorem little resembled deduction, understood as moving from a proposition to a proposition via logically valid transformations:

> On our view, in theoretical analysis one analyses the complexes of geometrical objects — their interrelations and interdependencies — involved in the proof of the desired theorem, not the deductive steps which would take us from the premises to the theorem (...). The steps of the analysis do not take us from one proposition to another, no matter what the direction of the relation of the logical consequence us which obtains between them, but from a geometrical object of a number of geometrical objects to another one.[15]

Thus, Hintikka and Remes observe that geometrical analysis requires an extensive reference to the geometrical objects themselves. Moreover, this usually involves a number of 'auxiliary constructions':

> (...) an analysis can succeed only if besides assuming the truth of the desired theorem we have carried out a sufficient number of

13 Literature on Euclid's

14 J. Hintikka, U. Remes, *The Method of Analysis: Its Geometrical Origin And Its General Significance*, Springer, Dordrecht 1974. The authors analyse Pappus' conception of analysis to be found in Pappus, *Collectionis quae supersunt*, F. Hultsch (ed.), vol. 3, Weidmann, Berlin 1876-78.

15 J. Hintikka, U. Remes, *The Method of Analysis, op. cit.*, p. 32.

> auxiliary constructions in the figure in terms of which the proof is to be carried out. In principle these may be carried out in the course of the analysis, too, but they can always be accomplished before it. This indispensability of constructions in analysis is a reflection of the fact that in elementary geometry an auxiliary construction (...) which goes beyond the *ekthesis* or the 'setting-out' of the theorem in terms of a figure, must often be assumed to have been carried out before a theorem can be proved. Hence, a proof cannot be found be means of analysis without such auxiliary constructions.[16]

This shows that in the Euclidean paradigm, pictures (such as found in Fig. 1 and Fig. 2) are *essential parts* of the geometrical proof. In other words, the validity of the proposition I.1 hangs together with the fact that one *sees* that the two circles do intersect at point C. This observation forces us to analyse in detail Kant's concepts of intuition and construction.

3. Intuition and construction

The basic idea behind Kant's conception of mathematical cognition is — as we have already observed — that mathematics is based on the pure intuitions of space and time. In an often quoted passage, Kant observes:

> Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyse and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts. But now let the geometer take up

16 *Ibidem*, p. 2.

> this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be draw at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question.[17]

It is easy to realize that Kant describes here the proof of the proposition 32 of Book1 of Euclid's *Elements*. Along the way, he claims that mathematical knowledge cannot be gained by recourse to concepts. The analysis of pure concepts cannot lead us to the establishment of any mathematical theorem — in order to prove anything, we need to *construct* our concepts, and for the construction we need intuition.

What is, then, intuition, and what is construction? All modes of cognition are deemed by Kant *representations*. Now, among representations he places *perceptiones*, which are "representations with consciousness." Perceptions can be further divided into sensations (*sensationes*) and cognitions (*cognitiones*). The former are perceptions "that refer to the subject as a modification of its state"; the latter are objective perceptions: intuitions or concepts. "[Intuition] is immediately related to the object and is singular; [concept] is mediate, by means of a mark, which can be common to several things." A concept which is pure, one "that has its origin solely in the understanding (not in a pure image of sensibility)", is called notion. One can also speak

17 B744.

of ideas, i.e. "concepts made up of notions, which goes beyond the possibility of experience".[18]

This classification is highly interesting for many reasons. First and foremost, it illustrates what are, according to Kant, the basic kinds of cognition. In his treatment of cognition he goes beyond his predecessors — Descartes, Leibniz and the British empiricists. For Descartes and his followers, cognition (or thinking) can be modelled as an analogue of seeing ideas with the 'mind's eye'. Moreover, the term 'idea' referred to anything what is an object of cognition, from figures and numbers, through the 'impressions' of physical objects, to pain and other emotions and feelings.[19] In Kant's conception, our mental machinery is much more complex. Although there exists an 'umbrella term' for all the objects of perception (*representationes*), there are clear functional divisions within this overarching category. From our perspective, the most important is the distinction between two kinds of *cognitiones*, i.e. objective perceptions: intuitions, which are singular, concrete and direct, and concepts, which simultaneously refer to a number of objects by means of a common mark, and as such are universal and indirect.

The crucial element of Kant's philosophy of mathematics is that mathematical cognition *must* be based on intuitions. He takes two different routes to substantiate this thesis: the metaphysical and the transcendental expositions. The *metaphysical exposition* begins with a characterization of space and time, which is followed by the description of the nature of mathematical judgements. This line of reasoning belongs to what Kant deems 'transcendental aesthetics', and consists of the following steps. First, space and time are not empirical concepts, for any

18 A320/B377.

19 Cf. I. Hacking, *Why Does Language Matter to Philosophy*, Cambridge University Press, Cambridge 1975, p. 27.

empirical cognition *assumes* space and time as the reference framework. "In order for certain sensations to be related to something outside me (...) the representation of space must already be their ground."[20] Similarly, "simultaneity or succession would not themselves come into perception if the representation of time did not ground them *a priori*."[21] Secondly, both space and time are *a priori*: it is impossible to represent that there is no space or time; they cannot be removed from our cognition. Thirdly, if so, space and time must be grounds for *a priori* and certain judgements, such as that only one straight line lies between two points.[22] Fourth, neither space nor time are discursive: they are not concepts that relate to things in general. They are pure, concrete intuitions, as one can represent only one space and one time. Different 'spaces' always result from a delineation of the one concrete space (pure intuition); similarly, "different times are only parts of one and the same time".[23] Finally, one can represent space and time only as given infinite magnitudes.

Thus, in the metaphysical exposition Kant considers first the nature of space and time and identifies a number of their features: they are non-empirical, *a priori*, concrete pure intuitions. Therefore, the judgements that relate to them, such as the geometrical propositions, must be *a priori*, certain and universal. This direction of inference is reversed in the transcendental exposition. Kant says:

> I understand by a transcendental exposition the explanation of a concept as a principle from which insight into the possibility of other synthetic *a priori* cognitions can be gained. For this aim it is required 1) that such cognitions actually flow from the given

20 A23/B38.

21 B46.

22 B39.

23 B47.

> concept, and 2) that these cognitions are only possible under the presupposition of a given way of explaining this concepts.[24]

In other words, Kant's strategy here is: to assume that "geometry is a science that determines the properties of space synthetically yet *a priori*"[25], and ask: "what then must the representation of space be for such a cognition of it to be possible?"[26] His answer is that space must be an intuition (to ground the syntheticity of the geometrical judgements), and one that is *a priori* (to ground the aprioricity of the geometrical propositions). This leads to the conclusion that space "has its seat merely in the subject, as its formal constitution of being affected by objects and thereby acquiring immediate representation, i.e., intuition, of them, thus only as the form of outer sense in general."[27] A similar line of reasoning applies to time.[28]

The transcendental exposition hangs together with the soundness of the assumption that mathematical propositions are necessary and universal (i.e., *a priori*), and yet synthetic. Kant devotes much attention to this problem. In his *Introduction* to the second edition of the *Critique of Pure Reason* he observes that his claims are contrary to a long tradition which takes mathematical judgements to be analytic. His famous defence of the syntheticity of pure mathematics begins by granting that mathematical propositions "are always *a priori judgements* and are never empirical, because they carry necessity with them, which cannot be derived from experience."[29] He proceeds to observe

24 B40.
25 *Ibidem*.
26 *Ibidem*.
27 B41.
28 B48-49.
29 B15.

that the proposition '7+5=12' cannot be thought of as merely analytic. The reason he gives is the following:

> The concept of twelve is by no means already thought merely by my thinking of that unification of seven and five, and no matter how long I analyse my concept of such a possible sum I will still not find twelve in it. One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or (...) five points, and one after another add the units of the five given in the intuition to the concept of seven. (...) The arithmetical proposition is therefore always synthetic. (...) Just as little is any principle of pure geometry analytic. That the straight line between two points is the shortest is a synthetic proposition. For my concept of the straight contains nothing of quantity, but only a quality. The concept of the shortest is therefore entirely additional to it, and cannot be extracted out of the concept of the straight line by any analysis. Help must here be gotten from intuition, by means of which alone the synthesis is possible.[30]

Thus, Kant's argument is that mathematical propositions are synthetic as they require representation in the intuition of space to check their validity. When one adds seven to five, one needs to *picture* this union; similarly, to check whether the straight line is the shortest line that connects two points, one needs to imagine this line (and perhaps some other lines connecting the given points). This shows that Kant follows the Euclidean paradigm at the deepest possible level: mathematics — both geometry and arithmetic — cannot dispense with 'pictures'. But the 'pictures' in question are not empirical objects as such objects fail to secure the certainty and universality of mathematical propositions. The 'pictures' are representations in pure intuition of space. Space

30 B15-16.

serves here as the environment for *a priori* constructions of mathematical concepts. More precisely:

> To construct a concept means to exhibit *a priori* the intuition corresponding to it. For the construction of a concept, therefore, a non-empirical intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition, but in both cases completely *a priori*, without having had to borrow the pattern for it from any experience. The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle.[31]

The quoted passage touches upon the most controversial aspect of Kant's philosophy of mathematics. As Kant puts it, "mathematical cognition considers the universal in the particular, indeed even in the individual (A714, B742)." Somehow, it is possible to justify the *universal* propositions of mathematics and to substantiate their *certainty*, by recourse to intuition, which is, *ex definitione*, concrete and individual. In other words, an intuitive construction of a particular, concrete triangle may serve as the basis for formulating propositions pertaining to each and every triangle. This is a place where Kant's philosophy of mathematics meets the philosophers' old friend — the problem of 'one

31 A713-14, B 741-42.

and many'. How is it possible that one concept (say, of a triangle) refers to a number of concrete things? Moreover, what is the foundation for claiming that each and every instantiation of a mathematical concept exhibits some necessary features of this concept? How to distinguish between the necessary and contingent features of a concrete object (e.g., a given triangle)? These questions lie at the heart of the controversies surrounding Kant's conception of mathematics.

4. Kant's *tertius homo*: between intuitions and concepts

The most important objection against the picture of mathematical cognition sketched above is phrased by Philip Kitcher in the following words:

> The inadequacy of pure intuition (...) stems immediately from the idea that we can, on inspection, determine the exact nature of a figure, whether physical or 'drawn in thought'. (...) The problem lies with the picture behind Kant's theory. That picture presents the mind bringing forth its creations and the naive eye of the mind scanning those creations and detecting their properties with absolute accuracy.[32]

Thus, according to Kitcher, there is a *mysterious* aspect to Kant's conception of mathematics. It is a *mystery* how concrete representations in intuition are related to concepts, which are universal. To be sure, intuitive representations are *instantiations* of concepts; however, one may ask what this instantiation boils down to. In other words, one is dealing here with the old problem of the relationship between universals and particulars. In his dialogue, *Parmenides*, Plato famously observed that in order to say that there is a resemblance between an abstract idea and a

32 P. Kitcher, *The Nature of Mathematical Knowledge*, Oxford University Press, Oxford 1985, p. 49-50.

concrete sensual object, one needs a third man or *tertius homo*, i.e. an independent standard, in virtue of which the similarity between the idea and the concrete object may be predicated. Kitcher's worry is that for Kant the similarity in question, i.e. the resemblance between a concept (say, of a triangle) and a concrete object (a triangle) given in intuition, is determined with the use of a mysterious faculty of mind (a 'naive eye') which is capable of detecting the essential properties of intuitive constructions with absolute accuracy. Moreover, the worry grows bigger once we take into account the fact that it is the concrete intuitive representations that are the ultimate foundations of our mathematical judgements and proofs. The 'passage' from a concept to its instantiation in intuition is crucial in both directions. On the one hand, it serves to identify concrete objects as triangles, cubes, etc. On the other, it is required to establish relationships between concepts and thus to prove theorems. The syntheticity of the mathematical knowledge and the very possibility of doing mathematics hangs together with it.

Kitcher's worry is legitimate, but it fails to acknowledge what Kant explicitly says. In the second part of the fragment of *The Critique of Pure Reason* quoted above (A714) one reads:

> The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent

Here, Kant speaks of the role of 'the action of constructing the concept'; it is in this action or procedure that the essential properties which bind together a concept and its representation in intuition are to be found. Kant's third man *is* this procedure:

> Now it is clear that there must be a third thing, which must stand in homogeneity with the category on the one hand and the appearance on the other, and makes possible the application of the former to the latter. This mediating representation must be pure (without anything empirical) and yet intellectual on the one hand, and sensible on the other. Such a representation is the transcendental schema.[33]

Kant viewed the theory of schemata as the most difficult, yet one of the most important parts of the *Critique*.[34] Indeed, the exposition of the conception of schemata is very dense: it takes a little more than one percent of the entire treatise. In face of that, Charles Sanders Peirce famously noted:

> [Kant's] doctrine of the schemata can only have been an afterthought, an addition to his system after it was substantially complete. For if the schemata had been considered early enough, they would have overgrown his whole work.[35]

Be that as it may, the conception of the schemata are central not only to Kant's philosophy of mathematics, but to his entire critical project. What, then, is a schema? Kant claims that schemata are purely formal conditions of sensibility, to which the use of concepts of understanding is restricted. Thus, every concept has a corresponding schema. Schemata are 'product of pure imagination', but of a special kind, distinguishable from images:

> (...) If I place five points in a row, ... , this is an image of the number five. On the contrary, if I only think a number in general,

33 A183/B177.

34 *Kants gesammelte Schriften*, Georg Reimer, Berlin 1900-, 18, 685-87; quoted after K.F. Jørgensen, *Kant's Schematism and the Foundations of Mathematics*, PhD Dissertation, Roskilde 2005, available online.

35 C.S. Peirce, *Collected Papers of Charles Sanders Peirce*, Vol. 1-8, Harvard University Press: Cambridge, Mass, C. Hartshorne, P. Weiss, and A. Burks (ed,), vol. I, p. 35; quoted after K.F. Jørgensen, *Kant's Schematism..., op. cit.*

> which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (e.g., a thousand), in accordance with a certain concept than the image itself, which in this case I could survey and compare with the concept only with difficulty. Now this representation of a general procedure of the imagination for providing concept with its image is what I call the schema for this concept.[36]

It is clear that, for Kant, schemata are procedures, i.e. general, non-singular rules or sets of such rules whose application enables one to *construct* objects corresponding to a given concept. Thus, Robert Hanna is mistaken, when he claims that "a schema is a quasi-objective exemplary or paradigmatic instance of a concept, produced by the pure imagination, such that it encodes the relevant conceptual content or conceptual information in a specifically spatial or temporal form."[37] If it were so, Kant's doctrine of the schematism would be susceptible to a variation of Kitcher's objection: one would need to posit a special faculty of mind capable of 'producing' such exemplary representations of a concept, and an additional faculty that would be capable of comparing a given intuitive object with the exemplar. Moreover, in many — if not most — cases such exemplary representations seem not to exist. Let us ask the following question: what would be an exemplary representation of a triangle? It is impossible to think of one. If any triangle given in the intuition is paradigmatic, then *every* such representation is. Kant realizes this when he says:

> No image at all would ever be adequate to the concept of a triangle in general. For it would never attain the universality of the concept, which makes it hold for all triangles, whether right-angled,

36 A140/B179.

37 R. Hanna, "Mathematics for Humans: Kant's Philosophy of Arithmetics Revisited", *European Journal of Philosophy*, 10 (2002), p. 346.

> acute-angled, and so on, but would always be limited to only a part of this sphere. The schema of the triangle can never exist anywhere but in thought, and it signifies a rule of synthesis of the imagination with respect to pure figures in space.[38]

The application of such a rule of synthesis, i.e. of a schema, is often invoked by Kant. For example, he says:

> We say that we cognize the object if we have effected synthetic unity in the manifold of intuition. But this is impossible if the intuition could not have been produced through a function of synthesis in accordance with a rule that makes the reproduction of the manifold necessary *a priori* and a concept in which this manifold is unified possible. Thus we think of a triangle as an object by being conscious of the composition of three straight lines in accordance with a rule according to which such an intuition can always be exhibited.[39]

The resulting picture is that of a schema as a universal, non-concrete rule, existing in thought, being a 'counterpart' of a concept, and operating fully necessarily and *a priori*. Lisa Shabel rightly observes that it is the Euclidean paradigm that constitutes "an interpretive model for the function of a transcendental schema."[40] Euclid's postulates, such as "To draw a straight line from any point to any point" or "To describe a circle with any centre and distance", may indeed be considered 'rules of thumb' or 'auxiliary tools' that are useful in the process of construction. But it is the process of construction of the relevant figures that is essential to the process of proving propositions. Such constructions are procedures *tout court*: any of Euclid's proofs is, in effect, a description of a process of construction which aims

38 A140/B180.

39 A105.

40 L. Shabel, *Mathematics in Kant's Critical Philosophy*, Routledge, London 2003, p. 109.

at justifying the proposition at hand. Now, following a procedure means following a set of *rules*. Recall the proof of Euclid's proposition I.32: "In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles." One begins here by *constructing* a triangle ABC, and then *producing* one side of it, BC, to D. Then, one *inspects* the angle ACD and *observes* that it equals the sum of the two interior and opposite angles CAB and ABC, as well as that the sum of the three interior angles of the triangle ABC, BCA, and CAB equals two right angles. Further one *draws* the line CE through the point C parallel to the straight line AB, and so on. It is clear that to proceed with such a proof means to follow a set of instructions or rules. Moreover, the resulting picture, either on paper or in imagination, is always of a concrete geometrical object. Yet, the conclusion, i.e. the proved proposition, is fully universal. The universality in question is guaranteed by the character of the rules used in the process of construction. Observe that these rules are not fully determinate: they do not require to draw a line of some precise length, or a triangle of some fully determined angles (e.g., a right-angled triangle).

In light of the above we cannot agree with Shabel who claims that "in the case of mathematical concepts (...) schemata are strictly redundant: no 'third thing' is needed to mediate between a mathematical concept and the objects that instantiate it since mathematical concepts come equipped with determinate conditions on and procedures for their construction."[41] On the one hand, this view clearly contradicts Kant's explicit statements, as when he says "give a philosopher the concept of

41 L. Shabel, "Kant's Philosophy of Mathematics", [in:] The Cambridge Companion to Kant and Modern Philosophy, P. Guyer (ed.), Cambridge University Press, Cambridge 2006, p. 111.

a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles."[42] On the other, Shabel's interpretation would lead directly to the recurrence of Kitcher's objection: if there were no schema coupled with the mathematical concepts — or if they were redundant — the concepts would apply directly to their instantiations in the intuition. But then one would need to postulate the existence of a special faculty of mind capable of grasping the essential properties of a concrete object, ones that correspond to the properties constituting the concept.

The above discussion centred around Kant's view of the geometrical practice. The thesis we embrace — together with a number of contemporary interpreters of Kant[43] — is that Kant takes Euclidean geometry as his paradigm of what mathematics amounts to. To this, there corresponds his doctrine of the schemata which 'mediate' between general concepts and concrete geometrical objects given in the intuition. The next question is how this conception fares *vis a vis* the disciplines of arithmetic and algebra.[44]

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42 A716/B744.

43 E.g., Lisa Shabel, Michael Friedman, K.F. Jørgensen.

44 In the above outline of Kant's doctrine of schemata we have left out a number of important problems. E.g., there is a heated debate as to the exact nature of mathematical intuition in Kant (see the exchange between Hintikka and Parsons and some commentators). One should also bear in mind that the doctrine of schematism is central not only to Kant's philosophy of mathematics, but to his entire critical project. E.g., he distinguishes between different kinds of schemata (empirical, geometrical, transcendental), and speaks of the schemata of magnitude, substance, cause, community (reciprocity), possibility, actuality, necessity, relation, etc. See, e.g., the papers collected in *Kant's Philosophy of Mathematics. Modern Essays*, C.J. Posy (ed.), Kluwer Academic Publishers, Dordrecht 1992.

5. Arithmetic and algebra

In his influential paper of 1941, "Kant's Theory of Mathematical and Philosophical Reasoning", C.D. Broad complains that Kant's conception of mathematics "was evidently made up primarily to deal with geometry, and was then extended forcibly to deal with arithmetic and algebra."[45] This view is highly plausible once we realize what 'happens' when we represent a number or an algebraic variable or formula in the intuition of space (and time). There seems to be no clear intuitive representations of a given number, say '7'. When one speaks of geometrical figures, one may say that to a concept of a triangle there correspond infinitely many concrete representations in the intuition. The number '7', however, seems to be a concept with no clear instantiations (surely, one can 'see' or 'construct' in the intuition seven apples, seven strokes, seven chairs, etc., but these are not pure representations of the *number* seven). The case is even worse with algebra. When one contemplates the formula '*a+b=c*', there is no immediate way of representing it in the intuition, apart from imagining the five symbols: 'a', '+', 'b', '=', and 'c' and putting them together. There seems to be no way to derive synthetic *a priori* algebraic judgements. There are, however, a number of assumptions standing behind Broad's pessimistic view of Kant's philosophy of arithmetic and algebra. The first is that arithmetic pertains to numbers which have nothing to do with space: they are not spatial objects. The second treats algebra as a kind of 'generalization of' or 'abstraction from' arithmetic: algebraists only replace concrete numbers with symbolic variables to establish fully abstract relationships between *arbi-*

45 C.D. Broad, "Kant's Theory of Mathematical and Philosophical Reasoning", *Proceedings of the Aristotelian Society*, 42 (1941), p. 5.

trary quantities. It seems, however, that Kant would reject both these assumptions.

Let us begin with arithmetic. Kant is quite explicit as to the role of the intuition of space in the arithmetical practice. In the already quoted passage he notes:

> The concept of twelve is by no means already thought merely by my thinking of that unification of seven and five, and no matter how long I analyse my concept of such a possible sum I will still not find twelve in it. One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or (...) five points, and one after another add the units of the five given in the intuition to the concept of seven. (...) The arithmetical proposition is therefore always synthetic.[46]

Thus understood, the representation of numbers requires both space and time. This claim may strike us as unintuitive. However, one needs to bear in mind that Kant had a different mathematical background than we have. In particular, it is argued that, like in geometry, in arithmetic he followed the Greek tradition. Daniel Sutherland gives the following rough exposition of the Greek concept of number:

> The Greek term for number, '*arithmos*', had different senses. The thinnest sense seems to have been nothing more than a particular collection of things, such as the number of sheep in a particular field or the number of shoes in my closet, in roughly the way we think of a particular set. This notion of *arithmos* presupposes a choice of unit, such as a shoe or pairs of shoes. Another closely related sense of *arithmos* was the number that resulted from enumerating the members of a collection, which is sometimes called the 'counting-number' of the counted collection. It, too, presupposes the choice of a unit, which is often called a 'counting-concept.' There is also a further conception of number apart from a

46 B15-16.

> kind of thing, one that is based on collections of 'pure' units. The motive for pure units may arise from the desire for a general representation of number apart from any kind of thing counted. The close connection of *arithmos* to particular collections of things and the notion of number as a collection of units were influential into the nineteenth century.[47]

There are two ideas pertaining to numbers which are worth noticing here. Firstly, the entire body of Greek mathematics is 'visual': both the geometrical figures and numbers are objects of 'grasping', which process is referred to with the verb *noein*. Initially, *noein* meant 'to see' (physical objects), but later its 'semantic field' was enriched to include such operations as 'to imagine', 'to grasp' or 'to think'. In a recent study, Reviel Netz scrutinizes the various uses of *noein* by Greek mathematicians, pointing out the pivotal role of this verb in the writings of Euclid, Archimedes or Pappus.[48] For example, Euclid, when presenting a proof for a proposition, asks us to 'conceive of', 'imagine' or 'construe' a point, or a line, or a triangle (in many of such cases the verb *noein* is utilized). Secondly, the concept of a number presupposes the notion of a unit and the possibility of identifying such units within a manifold one conceives. Aristotle notes, for example, that "plurality is as if it were a genus of *arithmos*; for *arithmos* is a plurality measurable by the one. And in some sense the one and *arithmos* are opposed, not as contraries, but (...) as some relative things are; for the one in so far as it is a measure is opposed to *arithmos* in so far as *arithmos* is measurable."[49]

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47 D. Sutherland, "Arithmetic from Kant to Frege: Numbers, Pure Units, and the Limits of Conceptual Representation", *Philosophy,* 83 (2008), p. 139.

48 R. Netz, "Imagination and Layered Ontology in Greek Mathematics", *Configurations,* 17 (2009), pp. 19-50.

49 *Iota*, (I)6, 1057a2-6.

Kant holds an analogous view, as evidenced in the following passage:

> (...) the pure schema of magnitude (*quantitas*), as a concept of the understanding, is number, a representation which summarizes the successive addition of one to one (homogeneous). Thus number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition.[50]

In order for arithmetic to work, Kant postulates, as in the case of geometry, the existence of relevant schemata. In particular, the schema corresponding to the concept of a number is *the schema of magnitude*. The schema works in such a way that, given some unit, it synthesizes, by 'successive addition of one to one', the parts of a manifold, which are qualitatively identical, but numerically different. Thus, it is possible to represent a given number (say, 7) as a manifold of seven dots, or strokes, etc. This would be impossible, however, if arithmetic operated merely at the level of concepts, because "according to mere concepts of the understanding, it is a contradiction to think of two things outside of each other that are nevertheless fully identical in respect of all their inner determinations (of quality and quantity); it is always one and the same thing thought twice (numerically one).[51]

It also follows that for the schema of magnitude to work, one requires *both* the intuitions of space *and* time. Kant notes:

> Time, as you correctly notice, has no influence on the properties of numbers (as pure determinations of magnitude). (...) The science of numbers, notwithstanding the succession that every construct of magnitude requires, is a purely intellectual synthesis, which we represent to ourselves in thought. But insofar as specific

50 A142–3/B182.

51 *Kants gesammelte Schriften..., op. cit.*, 20, 280; quoted after K.F. Jørgensen, *Kant's Schematism..., op. cit.*

> magnitudes (quanta) are to be determined in accordance with this science; and this grasping must be subjected to the condition of time.[52]

Frege accused Kant of formulating a theory of arithmetic that is incapable of accounting for our ability to use 'large' numbers, as it is impossible for such a number as 170,034 to be given in the intuition. However, Kant's view is the exact opposition of Frege's. He notes:

> The arithmetical proposition is therefore always synthetic; one becomes all the more distinctly aware of that if one takes somewhat larger numbers, for it is then clear that, twist and turn our concepts as we will, without getting help from intuition we could never find the sum by means of the mere analysis of our concepts.[53]

The interpretation of this surprising claim that we would like to offer is the following. On the one hand, Kant repeatedly stresses that it is impossible to conceive of the mathematical practice as based solely on concepts. Mathematics must be synthetic. On the other hand, one should notice that the essential feature of the schema of the magnitude is that it may operate with *different units*. Let us consider the passing of time: we can measure it by seconds, minutes, hours, days, months, years, decades, etc. While operating with the schema of magnitude, we may choose the appropriate unit, and in this way obtain some synthesis of the 'manifold of the passing time', which leads to the understanding of the given chronology of events. In connection to this, Kant observes:

52 *Kants gesammelte Schriften…, op. cit.*,10, 556-57, quoted after K.F. Jørgensen, *Kant's Schematism…, op. cit.*

53 B16.

> Now pure synthesis, generally represented, yields the pure concept of the understanding. By this synthesis, however, I understand that which rests on a ground of synthetic unity *a priori*; thus our counting (as is especially noticeable in the case of larger numbers) is a synthesis in accordance with concepts, since it takes place in accordance with a common ground of unity (e.g., the decade). Under this concept, therefore, the synthesis of the manifold becomes necessary.[54]

Now, let us ask what happens when one intends to add two large numbers, say 170,034 to 140,032. It is impossible to represent those numbers in intuition when 1 is our unit of choice. However, we may begin with a different unit, say 10000: this manoeuvre reduces the problem to adding 17 to 14, which, in turn, leads to the conclusion that the result of addition has 31 units (of 10000). Next, we may take advantage of the unit 1 (or 10), and finally arrive to the conclusion that the result of adding 170034 to 140032 is 310066.

Let us turn now to algebra. In our exposition of the Kantian conception of algebraic practice we shall follow an interpretation of Lisa Shabel.[55] Kant speaks of algebra in two passages of the *Critique of Pure Reason*. The first reads:

> But mathematics does not merely construct magnitudes (*quanta*), as in geometry, but also mere magnitude (*quantitatem*), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it chooses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, extraction of roots, etc. and, after it has also designated the general concept of quantities in accordance with their different relations, it then exhibits all the procedure through which mag-

54 B104.

55 L. Shabel, "Kant on the 'Symbolic Construction' of Mathematical Concepts", *Studies in History and Philosophy of Science*. 29 (1998), pp. 589-621.

> nitude is generated and altered in accordance with certain rules in intuition; where one magnitude is to be divided by another, it places their symbols together in accordance with the form of notation for division, and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves), which discursive cognition could never achieve by means of mere concepts.[56]

And the other is:

> Even the way algebraists proceed with their equations, from which by means of reduction they bring forth the truth together with the proof, is not a geometrical construction, but it is still a characteristic construction, in which one displays by signs in intuition the concepts, especially of relations of quantities, and, without even regarding the heuristic, secures all inferences against mistakes by placing each of them before one's eyes.[57]

The obvious problem with this conception is the following: Kant seems to claim that in algebra one operates with *symbolic constructions*, as when he says that an algebraist "places the symbols together in accordance with the form of notation for division, and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction." In other words, it seems that, in algebra, one constructs the equation '*a+b=c*' by exhibiting the symbols *'a', '+', 'b', '='* and *'c'* in intuition and by manipulating them gets a result. Such a reading, favoured by Hintikka and Parsons,[58] is troublesome. It is difficult to understand how such a manipulation of pure symbols can yield any comprehensible result. Moreover, this interpretation makes the entire Kantian conception of mathematics incoherent: there is no clear connection between geometry and

56 A717/B745.

57 A734/B762.

58 Cf. their papers in *Kant's Philosophy of Mathematics. Modern Essays, op. cit.*

arithmetic on the one side, and algebra on the other, apart from a very general similarity that all their objects are constructed in pure intuition.

A somewhat different interpretation is proposed by Michael Friedmann, who claims that for Kant arithmetic corresponds to the theory of numerical magnitudes in Euclid's *Elements* (books VII-IX), and algebra to the Euclidean theory of ratio and proportion (Book V). He claims further that there is a close relationship between arithmetic and algebraic constructions:

> There are actually two distinguishable, although closely related, aspects to symbolic construction. On the one hand, in finding the magnitude of anything we will employ the successive progression underlying the number series: either by generating a whole number or fraction in a finite number of steps or by generating an infinite approximation to an irrational number. On the other hand, however, successive iteration is also employed in the mere manipulation of signs in algebraic formulas: such 'operation of a calculus' is also an iterative, step by step procedure.[59]

Thus, Friedmann assumes that algebra is a certain *generalization* of arithmetic.

Lisa Shabel firmly disagrees with the above presented interpretations. She believes that they misidentify important presuppositions of Kant's theory of algebra. Firstly, for Kant algebra is *not* generalized arithmetic. Secondly, Shabel criticizes also the claim Kant speaks of symbolic construction as a necessary element of a mathematical proof or equation-solving. She says:

> This supposition implies that the arbitrary marks chosen to express a mathematical relationship algebraically in the form of an equation are necessarily involved in the solution of that equa-

59 M. Friedman, *Kant and the Exact Sciences*, Harvard University Press, Cambridge, Mass. 1992, pp. 119–120.

> tion, just as the figures of the geometer are, for Kant, necessarily involved in the demonstration of a geometric theorem. Moreover, it implies that the single symbol chosen to 'construct' an algebraic concept in intuition, say 'x', is somehow able to reveal more about that concept than the philosopher's discursive and unconstructed concept, which Kant assures us is incapable of leading to synthetic mathematical judgements.[60]

Shabel further claims that in order to provide a satisfactory interpretation of Kant's conception of arithmetic, one needs to consider the mathematical practice of his time. To do so, she reviews the mathematical textbooks of Christian Wolff, used by Kant when he taught mathematics in Koenigsberg. It turns out that Wolff does not consider algebra as a science of its own; rather, it is a method used *both* in arithmetic and geometry as an aid. This understanding of algebra follows the Cartesian idea of *constructing equations*. On Shabel's reconstruction, the method works as follows:

(1) One distinguishes the given or known magnitudes of the (geometrical or arithmetical) problem from those that are unknown by symbolizing the former by the first letters of the alphabet, and the latter by the last.

(2) One finds as many equations as there are unknown magnitudes sought; the equations relate the known and unknown magnitudes according to the given conditions of the problem.

(3) The equations are transformed, or solved simultaneously, so that all unknown magnitudes are expressed algebraically in terms of known magnitudes.

(4) The final solution of a geometrical problem solved algebraically amounts to the geometrical construction of the

60 L. Shabel, "Kant on the 'Symbolic Construction' of Mathematical Concepts", *op. cit.*, p. 598.

> magnitude represented by the symbol for the unknown quantity sought for in the problem. The same holds, *mutatis mutandis*, for arithmetical problems.[61]

Now, the question is how this all fits into the Kantian framework. Shabel stresses that Kant makes a distinction between a magnitude (*quantum*) and a mere magnitude (*quantitas*), as evidenced at the beginning of the fragment A717/B745 quoted above. The difference between them may be illustrated with the following example: when I construct a given triangle in the intuition, I construct a certain *quantum*, e.g. the area of the triangle. However, the area may also be represented as straight line segment: in this case we abstract "from the constitution of the object" and depict a mere magnitude. Now, according to Shabel, the letters of algebra symbolize such straight line segments, which stand for mere magnitudes (*quantitates*). For instance, "the algebraic expression $a \div b$ symbolizes the construction of the quotient of the magnitudes symbolized by a and b; that is, $a \div b$ symbolizes the geometrical construction of the fourth proportional x such that b:a::unit:x. The algebraic expression is short-hand for the sometimes tedious geometric construction of 'the objects themselves', in this case the straight line segments *a*, *b*, and *x*."[62]

On this reading, the symbols *a, b, x*, or ÷ are not objects constructed in the intuition; rather, they are short-hands for a performable construction. If so, one must ask why is Kant saying that what algebraist does "is still a characteristic construction, in which one displays by signs in intuition the concepts, especially of relations of quantities, and, without even regarding the heuristic, secures all inferences against mistakes by placing each of them before one's eyes." Shabel's answer is that in case of algebra Kant speaks of a different kind of 'construction':

61 *Ibidem*, p. 602-603.

62 *Ibidem*, p. 614.

> The sort of construction that, for Kant, justifies a mathematical demonstration and distinguishes mathematical from philosophical reasoning is pure, schematic, *and* ostensive. In the particular case that algebra is applied to the solution of a geometric or arithmetic problem, such a construction might be *symbolized* for the sake of (algebraic) argument, rather than actually carried out. The possibility of its being carried out, by the imagination in accordance with *a priori* concepts and certain rules, is what allows such a 'symbolic construction' to stand in for its ostensive referent, the 'object itself'.[63]

Thus, Shabel claims that the pure symbolic representation of certain relations is not a construction *sensu stricto*. However, each and every such 'symbolic construction' may at will be substituted with a proper (spatial or spatio-temporal) construction its stands for.

6. 'Corollaries': Infinity, Axiomatic Systems, Decidability

The above presentation of Kant's philosophy of mathematics makes it possible to draw three 'philosophical corollaries', concerning infinity, axiomatic systems and decidability.

First, as regards the notion of infinity, Kant has several things to say. First, he speaks of the infinity of space and time:

> Space is represented as an infinite given magnitude. Now one must, to be sure, think of every concept as a representation that is contained in an infinite set of different possible representations (as their common mark), which thus contains these under itself; but no concept, as such, can be thought as if it contained an infinite set of representations within itself. Nevertheless space is so thought (for all the parts of space, even to infinity, are simultane-

63 *Ibidem*, p. 616-617.

> ous). Therefore the original representation of space is an *a priori* intuition, not a concept.[64]

> The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation time must therefore be given as unlimited. But where the parts themselves and every magnitude of an object can be determinately represented only through limitation, there the entire representation cannot be given through concepts, 'for they contain only partial representations', but immediate intuition must ground them.[65]

The interesting aspect of the first passage is that Kant speaks of space as "an infinite *given* magnitude". It seems, therefore, that he suggests that space is an *actual* infinity. In other words, one may draw the conclusion that Kant believes in the possibility of representing the actually infinite in one's intuition. On closer reading, however, this interpretation must be rejected. Observe, firstly, that the quoted passages underline that any representation of space and time is a *limitation* of a larger space or time. Thus, Kant is not describing here a *visio infinitatis*, or a direct perception of the actual infinity. Rather, he points out that any representation of space and time *presupposes* that there is a larger space or time. Put differently: he speaks of a *potential* infinity. Secondly, Kant stresses that infinity cannot be comprehended through concepts alone, since "no concept, as such, can be thought as if it contained an infinite set of representations within itself" and "the entire representation cannot be given through concepts, 'for they contain only partial representations', but immediate intuition must ground them." Finally, one may ask, how to explain Kant's reference to space as "infinite *given*

64 B40.

65 B48.

magnitude". The answer is fairly simple: the term 'magnitude' Kant uses is not *quantum* (some concrete magnitude), but *quantitas* (mere magnitude). Therefore, he does not treat space and time as concrete objects which are infinite; rather, he speaks of a certain *aspect* thereof. In relation to concrete magnitudes Kants says: " A magnitude is infinite if none greater than it (i.e., greater than the multiple c of a given unit contained in it) is possible. Now no multiplicity is the greatest, because one or more units can always be added to it. Therefore an infinite given magnitude (...) is impossible."[66] At the same time he claims that "the true transcendental concept of infinity is, that the successive synthesis of units in measuring a *quantum*, can never be completed."[67] This may be interpreted as positing that a *quantitas* of a *quantum* is infinite if the schema of magnitude fails to determine the *quantum*. On this reading, space and time are infinite for one can never measure them with the use of the schema of magnitude. It does not mean, however, that space or time are *infinites in actu*: we never represent them in intuition as actually infinite. Space and time are only potentially infinite, and one may form a concept of potential infinity only because the schema of magnitude fails to 'measure' space (or time).

Our second 'philosophical corollary' concerns Kant's understanding of axioms and the role of axiomatic systems *vis a vis* his conception of mathematics. As we have already observed, Kant claims that there are axioms in mathematics, understood as self-evident statements based on the pure intuition of space. The interesting question reads, what is the role of the Kantian axioms? First, it is important to notice, that Kant proclaims the existence of axioms in geometry, but rejects arithmetical axioms. Axioms, according to him, must be both synthetic and general.

66 A430/B458.

67 A432/B460.

In arithmetic, however, the general claims are analytic: "for that equals added to or subtracted from equals give an equal are analytic propositions, since I am immediately conscious of the identity of one generation of a magnitude with the other; but axioms ought to be synthetic *a priori* propositions."[68] The synthetic propositions of arithmetic, on the other hand, are not general:

> '7 + 5 = 12' is not an analytic proposition. For I do not think the number 12 either in the representation of 7 nor in that of 5 nor in the representation of the combination' of the two (...). Although it is synthetic, however, it is still only a singular proposition. Insofar as it is only the synthesis of that which is homogeneous (of units) that is at issue here, the synthesis here can take place only in a single way, even though the subsequent use of these numbers is general. If I say: "With three lines, two of which taken together are greater than the third, a triangle can be drawn," then I have here the mere function of the productive imagination, which draws the lines greater or smaller, thus allowing them to abut at any arbitrary angle. The number 7, on the contrary, is possible in only a single way, and likewise the number I2, which is generated through the synthesis of the former with 5. Such propositions must therefore not be called axioms (for otherwise there would be infinitely many of them) but rather numerical formulas.[69]

Thus, Kant believes axioms are characteristic of geometry only.

Moreover, the role of axioms as elements of deductive systems is entirely alien to Kant's conception of mathematics. For him, a proof is always a construction, not a transformation of formulae according to some logical rules of inference.[70] Apart from embracing the Euclidean paradigm of mathematics, there is a deeper reason for this. As Michael Friedmann observes, the

68 A164/B205.

69 A164/B205.

70 Cf. B763.

Aristotelian logic Kant utilizes is *monadic*; as such, it is not sufficient to represent infinite mathematical structures. "Such infinite structures, in modern logic, are represented by the use of nested sequences of universal and existential quantifiers using polyadic logic. These same representations, from Kant's point of view, are instead made possible by the iterative application of constructive functions in the 'productive imagination'.[71]

To sum up: in Kant's view there exist axioms of geometry, but not of arithmetic. They are synthetic *a priori* propositions, which are general, i.e. they pertain to a class of representations, and not a singular representation. However, the role of axioms is severely limited. To be sure, there are *inferences* in mathematics, not only constructions. But in connection to this Kant comments:

> For as it was found that all mathematical inferences (*Schlüsse*) proceed in accordance with the principle of contradiction (...), it was supposed that the fundamental propositions of the science can themselves be known to be true through that principle. This is an erroneous view. For though a synthetic proposition can indeed be discerned in accordance with the principle of contradiction, this can only be if another synthetic proposition is presupposed, and if it can then be apprehended as following from this other proposition; it can never be so discerned in and by itself.[72]

Thus, on Kant's view, mathematical reasoning, even if sometimes proceeds by inferences based in the principle of contradiction, is always ultimately based on synthetic propositions (not necessarily axioms!), which, in turn, are justified by a construction in pure intuition.

71 M. Friedmann, "Kant on Geometry and Spatial Intuition", manuscript available online, p. 7.

72 B14.

Finally, let us tackle the problem of decidability. In an often quoted passage Kant says:

> It is not as extraordinary as it initially seems that a science can demand and expect clear and certain solutions to all the questions belonging within it (*quaestiones domesticae*), even if up to this time they still have not been found. Besides transcendental philosophy, there are two pure sciences of reason, one with merely speculative, the other with practical content: pure mathematics and pure morals.[73]

The justification for the decidability of all mathematics flows from the assumptions of Kant's conception. Out of their very nature, mathematical object are schematisable and thus constructible within the finite amount of time. Therefore, it is in principle possible to completely determine every mathematical object and, what follows, to determine the truth value of any mathematical proposition.

There is one problem with this conclusion. According to some commentators, Kant distinguishes between the construction of a mathematical object and the postulate of its existence.[74] On this reading, one may postulate the existence of anything that is logically consistent, but may construct only what is constructible in the pure intuitions of space and time. It follows, *inter alia*, that — contrary to the claims of Bertrand Russell and others — Kant's philosophy of mathematics is not incompatible with the development of non-Euclidean geometries: they may be treated as postulated, but they cannot be subject to construction. It follows also that one can distinguish between two kinds of mathematics: the constructible and fully decidable, and the merely postulated, unconstructible and undecidable. This interpreta-

73 A480, B508.

74 Cf. R. Murawski, *Filozofia matematyki. Zarys dziejów*, 2nd edition, Wydawnictwo Naukowe PWN, Warszawa 2001, p. 54.

tion is defensible, but with two *provisa*. First, it alters the Kantian understanding of mathematics as the science establishing *synthetic* propositions. Second, it cannot be expressed in the conceptual scheme available to Kant. One may say that Euclidean geometry, as well as a bunch of non-Euclidean geometries, are sets of propositions depicting possible structures of space, and it is the Euclidean conception that is 'picked out' by pure intuition. However, "Kant has no notion of possibility on which both Euclidean and non-Euclidean geometries are possible. (...) Kant's conception of logic is not that of modern quantification theory, and he can have no notion like our of all logically possible structures."[75]

7. The mathematical subject extracted

The above sketched Kantian philosophy of mathematics may serve as the basis for describing his conception of the mathematical subject, i.e. a subject that is capable of 'doing mathematics': proving theorems, carrying out calculations, solving equations, etc. More exactly, the following description aims at identifying the minimal requirements that such a subject must meet.

Our description of the Kantian mathematical subject consists of the following theses:

(KS_1) The mathematical subject is *transcendental*.

As we have observed above, the term 'transcendental' refers to the necessary conditions of every possible cognition. Moreover, Kant points out that "the inner and sensible intuition of our mind (...) is not the real self as it exists in itself, or the transcendental subject, but only an appearance of this to us unknown

75 M. Friedmann, "Kant's Theory of Geometry", [in:] *Kant's Philosophy of Mathematics. Modern Essays, op. cit.*, p. 206, 207.

being, which was given to sensibility. The existence of this inner appearance, as a thing thus existing in itself, cannot be admitted, because its condition is time, which cannot be a determination of any thing in itself."[76] To put it differently, the transcendental ego cannot be given as an object of cognition. This opens up the way to various interpretations of this notion. The 'thinnest' reading, rather un-Kantian, identifies the transcendental subject with the set of propositions describing the necessary requirements for any act of cognition to emerge. On this minimal interpretation Kant's mathematical subject is no subject at all: speaking of the 'transcendental ego' is only a *facon de parler*, disguising the real goal of Kant's project. A more substantial reading is to treat the transcendental subject as a *fiction*: it brings together all the cognitive capabilities of the possible empirical subjects. In other words, the transcendental ego does not exist, but is a theoretical construction of a subject that is capable of perceiving everything that *any* empirical subject can perceive. Finally, on the strongest interpretation the transcendental subject does exist in the sphere of *noumena*, and the particular acts of cognition are *manifestations* of its activity. The textual evidence provides a foundation for both the second and third interpretations. Kant explicitly claims that the existence of the transcendental subject cannot be admitted at the level of the theoretical (pure) reason. However, he also claims that this existence can be postulated within the practical reason. Irrespective of which of the interpretation we choose (the second or the third), the important fact to underline is that the transcendental subject is capable of every possible mathematical cognition — it stands *vis a vis* the entire body of the possible mathematical knowledge. In this, it differs from any concrete, empirical subject, which — e.g., due to time limitations — cannot have complete mathematical knowledge.

76 A492.

(KS_2) The structure of the mathematical subject includes pure intuitions of space and time.

Kant's philosophy of mathematics follows the Euclidean paradigm, in which the whole of mathematics is ultimately based on spatial relationships. In order to secure the aprioricity and syntheticity of the mathematical propositions, Kant posits that non-empirical space and time are elements of the structure of the transcendental ego. He claims, further, that geometry is grounded on the intuition of space, while arithmetic is founded on both space and time. Thus, one may say that the propositions of mathematics are descriptions of the spatio-temporal structure of the transcendental subject. In this way, Kant rejects three claims: that mathematics is ultimately based on our sensual experience; that it is fully analytic (or devoid of meaning); and that it grasps some kind of platonic entities.

(KS_3) The mathematical subject is capable of *perceiving* mathematical objects in the pure intuitions of space and time.

The notion of 'perceiving' we use here may well be substituted with 'seeing' or 'imagining'. Thus, we would like to make a bold claim that according to Kant the cognition of the mathematical objects in pure intuition is not 'discursive' or 'dialectical'. Rather, it *resembles* seeing. In this, Kant follows a long tradition which begins with the utilization of the Greek word *noein* to refer to the process of mathematical cognition. In consequence, the perceived mathematical objects are always *concrete and singular*. One never 'sees' a triangle in general, but always a concrete triangle; one never perceives number or number seven in general, but always seven objects, identified with the use of the schema of magnitude (with the assumption of an arbitrary unit). Moreover, it follows that mathematical cognition is always *substantive*: one never contemplates signs or symbols devoid of meaning. At the fundamental level mathematical practice has meaning — it is not a 'free play of symbols'.

(KS_4) The mathematical subject *constructs* mathematical objects in pure intuition with the use of *a priori* rules or procedures, which are called *schemata*.

This is the key element of Kant's conception of mathematics. The transcendental subject not only perceives mathematical objects, but actively *constructs* them. This construction is rule-guided: in order to construct a triangle one follows a set of instructions which prescribe drawing a figure consisting of three lines in which any two lines taken together are greater than the third. Such an instruction, or *schema* in Kant's terms, serves an important function. It is general (it does not exhibit the features of a concrete object), yet its utilization leads to a singular construction. In this way, a schema is an intermediate between concrete representations in the intuition and generally applicable concepts. Schemata are not only general, they are also *a priori*: they are functions of pure reason. It is noteworthy that the conception of schemata exhibits the *practical* aspect of Kant's philosophy of mathematics. Mathematics, in a sense, is a result of a certain practice: the practice of constructing objects in pure intuition according to the rules of the appropriate schemata.

(KS_5) The mathematical subject is capable of forming universal concepts, which correspond to a multitude of concrete objects.

For Kant, knowledge is conceptual. Thus, mathematical knowledge must be expressible in concepts. The problem is that concepts are universal, i.e. they correspond to many objects simultaneously, while at the fundamental level one perceives concrete, singular mathematical objects. The possibility of forming universal concepts on the grounds of singular representations is conditioned on the existence of schemata. To put it differently: to every concept (say, of a triangle) there is connected a schema or a set of instructions that enables one to construct a particular instantiation of the given concept. As the schemata do not determine fully the features of the concrete object, they 'encode' the information as to what is conceptually essential in a singular intuitive construction.

(KS_6) The mathematical subject can link together concepts to form propositions in the general form *S* is *P*.
For Kant, who in logic follows in the footsteps of Aristotle, the most general form of a proposition is *S is P*. Thus, in order to form a proposition one needs to link together two concepts, denoted by the predicate letters *S* and *P*. To frame such a linking, one needs to construct — via the appropriate schemata — both concepts in the intuition and 'see' whether they are related in the way represented by the *S is P* proposition.

(KS_7) The mathematical subject is capable of making logical inferences.
Naturally, once propositions are formed, the transcendental subject may use them in logical inferences. However, the role of logical operations in Kant's philosophy of mathematics is significantly limited, especially when compared to more recent conceptions. Firstly, the logic Kant uses is very weak — it is, in essence, the syllogistic logic of Aristotle. What follows, and secondly, mathematicians make logical inferences, but it is not the basic mode of mathematical reasoning. Such a primary mode is construction in intuition (where no logic is used), while logical inferences, whose premises are always based on some intuitive constructions, play only a secondary or subsidiary role. In consequence, axioms are of little significance in Kant's philosophy

of mathematics. In particular, the idea of an axiomatic system is alien to the spirit of the *Critique of Pure Reason*.

(KS_8) The mathematical subject can decide any mathematical problem.
Given the structure of the transcendental ego, and in particular the fundamental role of pure constructions in solving mathematical problems, there are no undecidable questions in mathematics. The Kantian mathematical subject is, in principle, capable of determining the truth value of any mathematical proposition. One needs to bear in mind, however, that this is true of the *transcendental* subject, not an empirical one.

(KS_9) The mathematical subject is capable of *denoting* mathematical constructions with *symbols*.
On Kant's conception of algebra, one may denote any mathematical object (construction) with a symbol. It must be stressed, however, that there are no symbolic constructions *sensu stricto*. The algebraist's manipulation of symbolic formulae does not resemble a construction of a mathematical object in pure intuition. Rather, symbols are short-hands for full mathematical constructions. They may facilitate the mathematical argument; however, behind any symbolic expression there always stands a full-blooded, executable mathematical construction.

(KS_10) The mathematical subject is capable of forming the notion of the potential infinity, but not of the actual infinity.
According to Kant, one cannot speak of actual infinity, as it is not possible to construct an infinite object in the intuition. However, it is possible to form the notion of potential infinity. In this case, we are referring not to some particular possible construction, but rather to the *failure* of the schema of magnitude to measure the *quantitas* of space and time. Thus, Kant treats potential infinity as 'in-finity', as it may be understood as the impossibility of

ascribing some finite number of units to the pure intuitions of space and time.

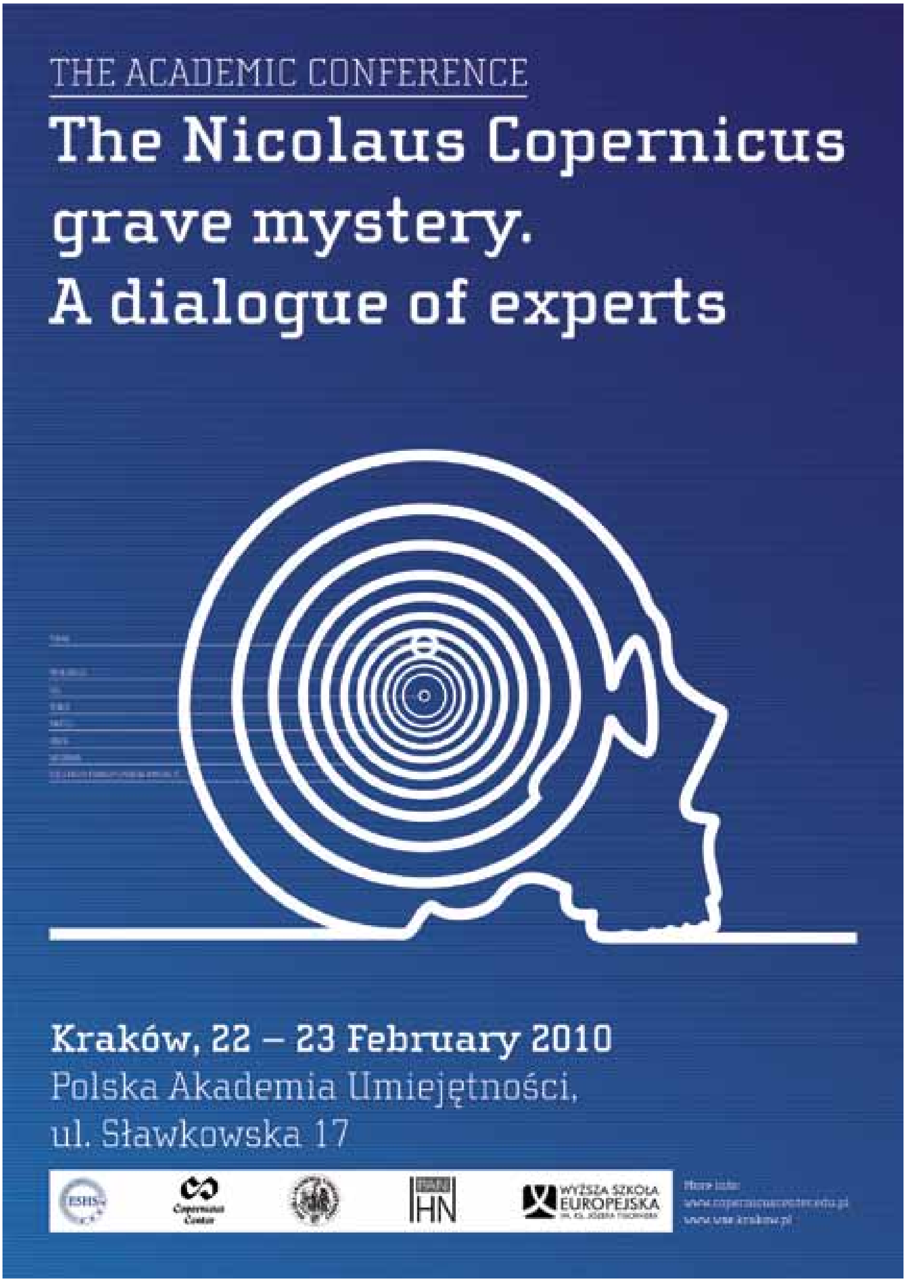
The *Nicolaus Copernicus Grave Mystery conference*, Kraków 22-23 February 2010. Photo: Adam Walanus

Michael Heller delivers a lecture within the *Science and the Big Questions* series, Kraków, 10 March 2010. Photo: Adam Walanus

The *Road to Reality with Roger Penrose*, 14th Kraków Methodological Conference, Kraków, 20-21 May 2010. Photo: Adam Walanus

Sir Roger Penrose delivers 2010 Copernicus Center Lecture, *Aeons Before the Big Bang*. Photo: Adam Walanus

The 2010 Copernicus Center Lecture, Kraków, 20 May 2010. Photo: Adam Walanus



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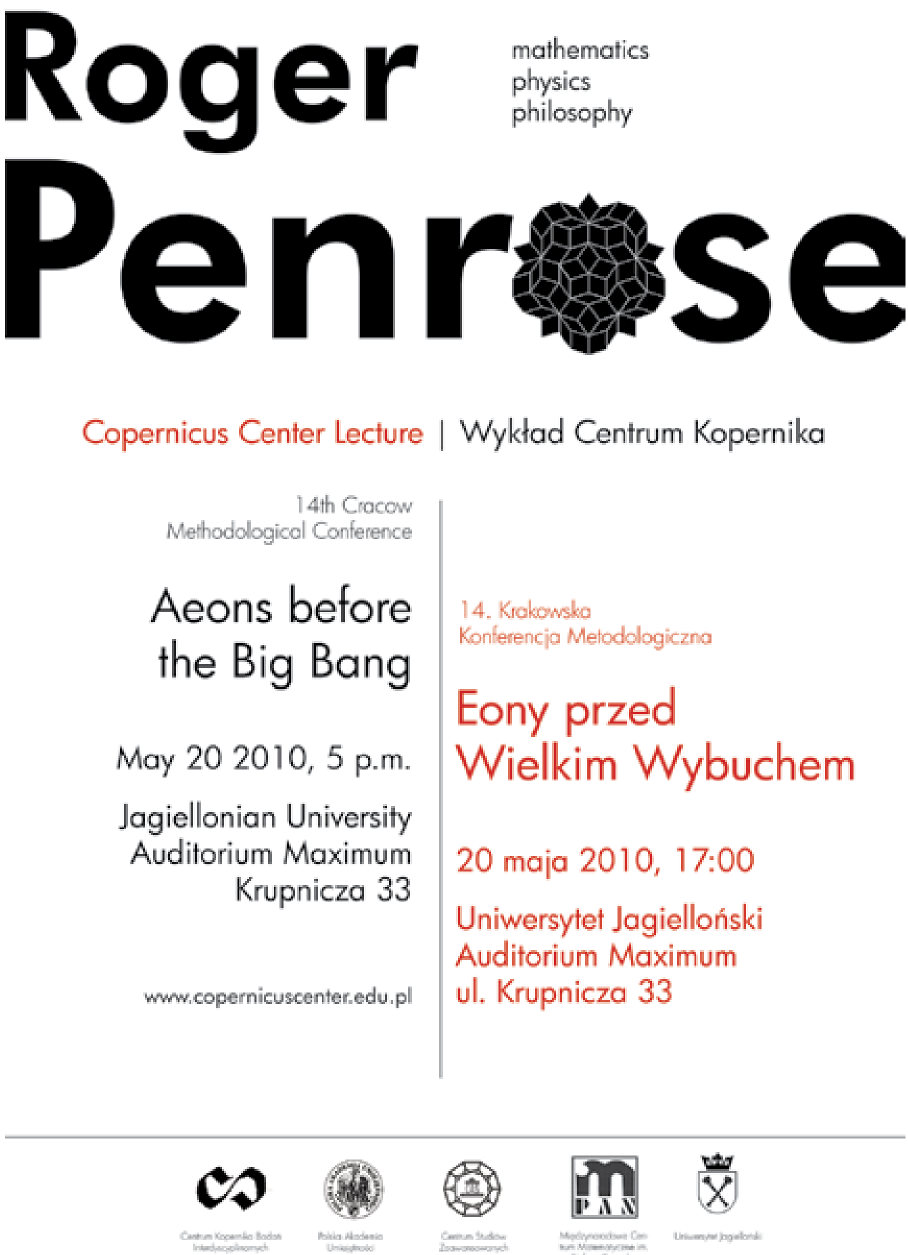

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